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Parameterisation of the Surface Fluxes

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PARAMETERISATION OF THE SURFACE FLUXES

By

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1. Introduction

While testing various parameterisation schemes for the surface fluxes, I came across some difficulties using the Monin-Obukhov similarity theory. Since I was following the method suggested by Busch et al. (1976) who did not report any such difficulty, I examined the problem thoroughly. This paper is a report of my findings.

In section 2 I briefly review the similarity theory as applied to the fluxes in the surface layer.
In section 3 I describe the numerical computation of the fluxes and discuss the results.
In section 4 I propose a parameterisation method which agrees with the similarity law, but which is very much simpler to use.

2. Similarity theory

We define the surface layer as the layer of the atmosphere near the ground where the wind velocity u and the potential temperature of the air θ can vary rapidly with height, but where the fluxes of heat and momentum do not change much from their ground values. We assume that the velocity and temperature profiles depend only on the following external parameters : height z, heat flux at the ground $\overline{w'\theta'}$, momentum flux at the ground $\overline{w'u'}$ and expansion coefficient of the air $\alpha = \frac{g}{\theta}$, and not on other parameters such as the Coriolis force or the height of the boundary layer. According to the similarity law we can then say that, when properly non-dimensionalised, the internal variables of the flow (velocity, temperature gradients) are universal functions of all the non-dimensional combinations of the external parameters.

Let us introduce the following scaling parameters :

$$u_* = \sqrt{|\overline{w'u'}|} \qquad (1)$$

$$\theta_* = -\overline{w'\theta'}/u_* \qquad (2)$$

insert

The only non-dimensional combination of the external parameters is a stability parameter :

$$\zeta = \frac{kg\ z\ \theta_*}{\theta\ u_*^2} = \frac{z}{L} \qquad (3)$$

where k, von Karman's constant, has been introduced to be consistent with most authors, and $L = \frac{\theta \, u_*^2}{kg \, \theta_*}$ is the Monin–Obukhov length. Hence we can write

$$\frac{kz}{u_*} \frac{\partial u}{\partial z} = \phi_m \, (\zeta) \qquad (4)$$

$$\frac{kz}{\theta_*} \frac{\partial \theta}{\partial z} = \phi_h \, (\zeta) \qquad (5)$$

The flux-profile relationships $\phi_m \, (\zeta)$ and $\phi_h \, (\zeta)$ can be determined by experiment. Empirical analytical formulae have been proposed for them. We will use the expressions suggested by Businger et al. (1971) :

$$\phi_m \, (\zeta) = \begin{cases} (1-\gamma_m \zeta)^{-\frac{1}{4}} & \text{(unstable conditions)} \quad (6.a) \\ 1+\beta\zeta & \text{(stable conditions)} \quad (6.b) \end{cases}$$

$$\phi_h \, (\zeta) = \begin{cases} R(1-\gamma_h \zeta)^{-\frac{1}{2}} & \text{(unstable conditions)} \quad (7.a) \\ R(1+\frac{\beta}{R} \zeta) & \text{(stable conditions)} \quad (7.b) \end{cases}$$

with $\gamma_m = 15$, $\gamma_h = 9$, $\beta = 4.7$, $R = 0.74$.

Using (6) and (7), (4) and (5) can be integrated analytically (see Paulson (1970), and Barker & Baxter (1975)), and we get:

$$u = \frac{u*}{k} \left[\ln \frac{z}{z_o} - \psi_m \left(\frac{z}{L}\right) + \psi_m \left(\frac{z_o}{L}\right) \right] \qquad (8)$$

$$\Delta\theta = \theta - \theta_o = \frac{\theta*}{k} R \left[\ln \frac{z}{z_o} - \psi_h \left(\frac{z}{L}\right) + \psi_h \left(\frac{z_o}{L}\right) \right] \qquad (9)$$

where z_o is the roughness length (height at which u = o), θ_o is the surface temperature, and

$$\psi_m(\zeta) = \int_o^\zeta \frac{1-\phi_m \, (\xi)}{\xi} \, d\xi$$

$$= \begin{cases} \ln \left[\left(\frac{1+x}{2}\right)^2 \left(\frac{1+x^2}{2}\right) \right] - 2 \arctan x + \frac{\pi}{2} \\ \quad \text{with } x = (1-\gamma_m \,)^{\frac{1}{4}} \quad \text{(unstable)} \quad (10.a) \\ -\beta\zeta \quad \text{(stable)} \quad (10.b) \end{cases}$$

$$\psi_h(\zeta) = \int_o^{\zeta} \frac{1-\frac{1}{R}\,\phi_h(\xi)}{\xi}\,d\xi$$

$$= \begin{cases} \ln\left|\left(\frac{1+y}{2}\right)^2\right| \text{ with } y = (1-\gamma_h\,\zeta)^{\frac{1}{2}} & \text{(unstable)} \quad (11.a) \\ -\frac{\beta}{R}\,\zeta & \text{(stable)} \quad (11.b) \end{cases}$$

In (8) and (9), $\psi_m(z_o/L)$ and $\psi_h\,(z_o/L)$ are often neglected. We keep them here in order to be completely general. (They are important for high instability).

We now have three equations : (3), (8) and (9), which relate the surface fluxes (u_*^2 and $u_*\,\theta_*$) to the quantities present in the model (u at the height z, and $\Delta\theta$) through the stability parameter ζ(or its dimensional equivalent L).

3. Computation of the fluxes

3.a Numerical methods

We want to solve (3), (8) and (9) for u_* and θ_*. Let us re-write these equations :

$$u_* = uk/\left[\ln \frac{z}{z_o} - \psi_m\left(\frac{z}{L}\right) + \psi_m\left(\frac{z_o}{L}\right)\right] \qquad (12.a)$$

$$\theta_* = \Delta\theta k/R \left[\ln \frac{z}{z_o} - \psi_h\left(\frac{z}{L}\right) + \psi_h\left(\frac{z_o}{L}\right)\right] \qquad (12.b)$$

$$L = \frac{z}{\zeta} = \frac{\theta u_*^2}{kg\ \theta_*} \qquad (12.c)$$

Because of the complicated analytical form of the functions ψ , it is not possible to substitute (12.c) into (12.a) and (12.b) and solve the resulting system for u_* and θ_* analytically.

Busch et al (1976) solved this system by successive substitutions, i.e. making a first guess for L they computed u_* and θ_* by using (12.a) and (12.b), substituted into (12.c) to compute a new guess for L and proceeded in this manner until the solution converges. I tried this method for various values of u and $\Delta\theta$, and several values of z_o. I found that, in general, it converged after 5 or 6 iterations, but that for some small values of u, it did not converge at all.

I then tried the Newton-Raphson method to solve the system of equations (12). It converged more rapidly than the previous method (3 or 4 iterations), but did not converge either in those cases where the first method failed.

3.b Discussion of the results

An example of this computation is shown in Fig. 1, where u_* and θ_* are computed in terms of u and $\Delta\theta$ for Z = 20m and Z_o = 0.01m. The computation does not converge in the hatched area.

In order to understand why the method does not converge in some part of the (u, $\Delta\theta$) plane, let us also show a graph of ζ in terms of u and $\Delta\theta$ (Fig. 2).

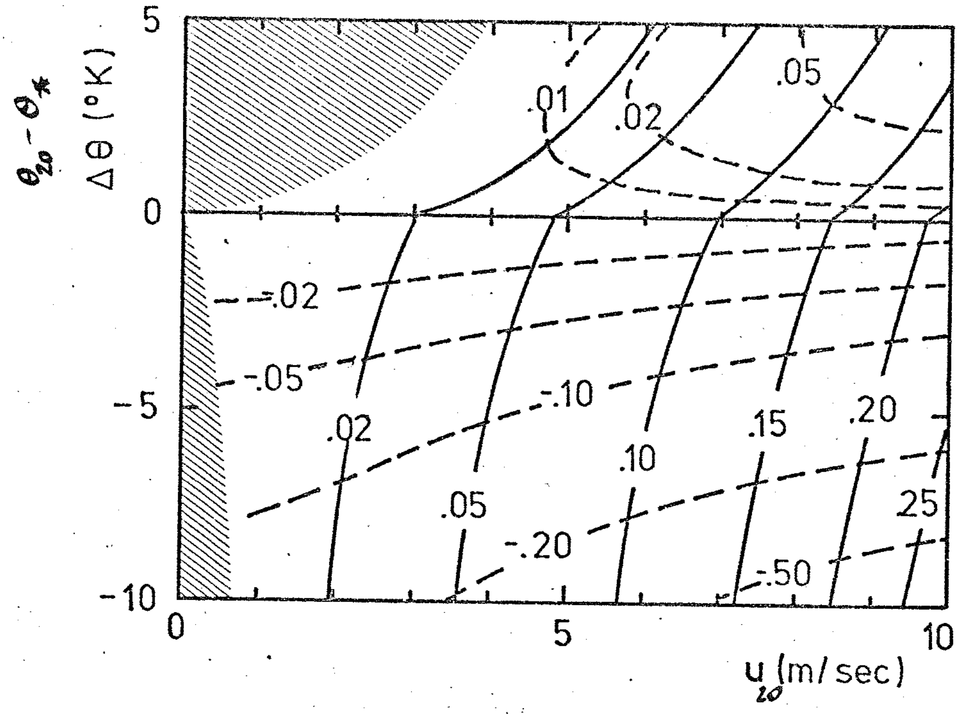

Fig. 1. Fluxes in terms of u and Δθ , for $\frac{z}{z_o} = 2x10^3$

——————— Momentum flux

-------- Heat flux

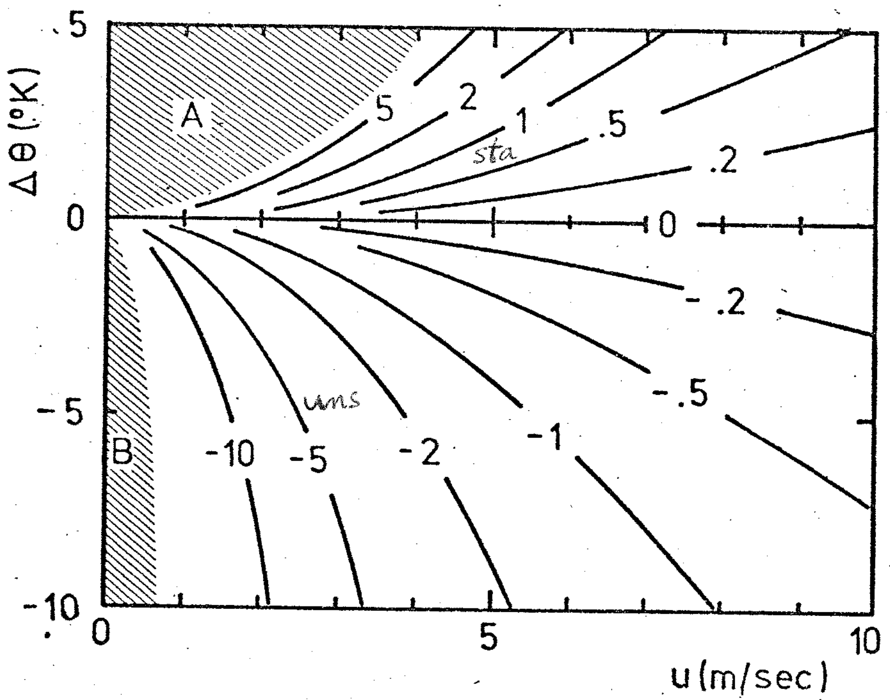

Fig. 2. Stability parameter $\zeta = \frac{kgz\ \theta_*}{\theta\ u_*^2}$ in terms of u and Δθ for $\frac{z}{z_o} = 2x10^3$

It can be seen that the iterative methods do not converge in the regions where ζ is very large, either positive or negative. One should emphasize that these regions are outside the area where experimental data exist to check the theory. The data published by Businger et al (1971) are for $-2.5 < \zeta < 2$. The reasons why the theory fails for small wind velocity are different in the stable and unstable cases. Let us look first at the stable region. Near the limit of the zone of non-convergence (area A in Fig. 2) we have

$$\zeta \to +\infty$$

It is easy to show that there is a direct relationship between ζ and the Richardson number :

$$Ri = \frac{g}{\theta} \; \frac{\partial\theta/\partial z}{(\partial u/\partial z)^2}$$

Using (3), (4) and (5) we obtain :

$$Ri = \zeta \frac{\phi_h(\zeta)}{[\phi_m(\zeta)]^2} \tag{13}$$

From the expression (6.b) and (7.b) it is easy to see that for $\zeta \to +\infty$, $Ri \to 1/\beta = 0.21$. This value is close to the generally accepted value of the critical Richardson number (0.25) beyond which the flow becomes laminar. Hence in region A the theory is not valid because the flow is no longer turbulent. The fluxes can be safely put to zero in this region.

The area B in Fig. 2 corresponds to the region of free convection : high instability and low wind. There is a singularity for $u = 0$ since, in this case, the momentum flux u_*^2 vanishes while the heat flux $u_* \theta_*$ should remain finite, implying

$$\theta_* \to -\infty$$

Close to the axis the problem becomes ill-conditioned and the iterative process does not converge.

4. Simplified formulation of the surface fluxes

4.a Use of a bulk Richardson number

For use in a forecast model the computation described above suffers from two important defects. First of all it is expensive in terms of computing time : It is an iterative computation and it also involves several functions (logarithm, arctangent) which are slow to compute. The second difficulty is that the method does not converge for some values of u and $\Delta\theta$. It may be that the model will never enter these regions, but one must take the possibility into account.

In order to get around the need for an iteration procedure in the model, one can do computations of the fluxes, as we have done, for a wide range of u, $\Delta\theta$ and Z_o, then either store the results in a table or try to fit an analytical three-dimensional function of u, $\Delta\theta$ and Z_o to them. If we can find a simple analytical function, this latter method is the best. Before we do this, however, let us make a slight modification to the method.

Let us define a bulk Richardson number:

$$Ri_B = \frac{gz\ \Delta\theta}{\theta\ u^2}$$

Using (3), (8) and (9) we can write directly :

$$\zeta = Ri_B \frac{\left[\ln\frac{z}{z_o} - \psi_m(\zeta) + \psi_m\left(\frac{z_o}{z}\zeta\right)\right]^2}{R\left[\ln\frac{z}{z_o} - \psi_h(\zeta) + \psi_h\left(\frac{z_o}{z}\zeta\right)\right]} \qquad (14)$$

This is an implicit relationship between Z_o, ζ and Ri_B. It can be solved for ζ in the stable case (Barker and Baxter, 1975) but not for the unstable case. Formally, however, we can write :

$$u_*^2 = u^2\ F\left(\frac{z}{z}\ ,\ Ri_B\right) \qquad (15.a)$$

$$u_*\ \theta_* = u\Delta\theta\ G\left(\frac{z}{z_o},\ Ri_B\right), \qquad (15.b)$$

where F and G are now the drag coefficient for momentum and heat respectively. The advantage of these expressions over (12) is that Ri_B is an explicit function of the model variables u and $\Delta\theta$, whereas in (12) L is a function of u_* and θ_* making the expressions implicit in u and $\Delta\theta$.

The functions F and G can be computed numerically, using the iteration procedure outlined above. Figures (3) and (4) show F and G respectively, in terms of Ri_B, for different values of z_o .

4.b Curve fitting

The problem is now to find simple analytical formulae for F and G. Let us first look at the behaviour of F and G for small Ri_B.

From (14) we can write :

$$\zeta = Ri_B \frac{\ln \frac{z}{z_o}}{R} \quad \text{for } \zeta \ll 1.$$

Making a limited expansion of ψ_m and ψ_h and retaining only the first order terms in ζ^m, we then have, using (8) and (9) :

$$u_* = uk / \left(\ln \frac{z}{z_o} + a_m \frac{\ln \frac{z}{z_o}}{R} Ri_B\right) \qquad (16.a)$$

$$\theta_* = k/R\left(\ln \frac{z}{z_o} + a_h \frac{\ln \frac{z}{z_o}}{R} Ri_B\right) \qquad (16.b)$$

and, finally, for Ri_B small :

$$F = \frac{k^2}{(\ln \frac{z}{z_o})^2} \left(1 - b_m Ri_B\right) \qquad (17.a)$$

$$G = \frac{k^2}{R(\ln \frac{z}{z_o})^2} \left(1 - b_h Ri_B\right) \qquad (17.b)$$

For Ri_B = o we find again the well-known logarithmic law. The coëfficients b_m and b_h can be determined by using the polynomial expansion suggested by Businger, et al. (1971) to fit the data near ζ= o.
We get b_m = 8.1 and b_h= 9.5.

Let us look now at the very unstable, or free convection case. There we want u_*^2 = o but $u_* \theta_*$ finite. From (15.b) we see that for u →o, G must behave like 1/u, i.e. like $|Ri_B|^{\frac{1}{2}}$. We suggest then the following expression for the unstable cases:

$$\frac{u_* \ \theta_*}{u \ \Delta\theta} = G\left(\frac{z}{z_o},\ Ri_B\right) = \frac{a}{R}\left(1 - \frac{b_h \ Ri_B}{1+C_h\left(\frac{z}{z_o}\right)\sqrt{|Ri_B|}}\right) \qquad (18)$$

with $a = \left[k/\ln\left(\frac{z}{z_o}\right)\right]^2$.

We can now use dimensional analysis to find the form of $C_h\ (z/z_o)$. In the case of free convection the only external parameters of the flow are the heat flux $\overline{w'\theta'}$, the coefficient of expansion α and the height z.

The scaling temperature is now :

$$\theta_* = \overline{w'\theta'}^{\,2/3}\ \alpha^{-1/3}\ z^{-1/3} \qquad (19)$$

Hence if the similarity theory holds for free convection we can write :

$$\frac{z}{\theta_*}\frac{\partial\theta}{\partial z} = C \qquad (20)$$

where C is a constant. Substituting (19) into (20) and integrating from z_o to z, we get :

$$\Delta\theta = C\ \overline{w'\theta'}^{\,2/3}\ \alpha^{-1/3}\ (z^{-1/3} - z_o^{-1/3})$$

or $$\overline{w'\theta'} = u_*\theta_* = C'\alpha^{\frac{1}{2}}|\Delta\theta|^{3/2} \Big/ \left(z^{-1/3} - z_o^{-1/3}\right)^{3/2} \qquad (21)$$

Let us now write (18) for $u \to o$ (i.e. $Ri_B \to -\infty$)

$$u_*\theta_* = -\frac{a\ b_h}{C_h\left(\frac{z}{z_o}\right)}|\Delta\theta|^{3/2}\ \alpha^{\frac{1}{2}}\ z^{\frac{1}{2}} \qquad (22)$$

Comparing (21) and (22) we find :

$$C_h\left(\frac{z}{z_o}\right) = -\frac{a\ b_h z^{\frac{1}{2}}\ (z^{-1/3} - z_o^{-1/3})^{3/2}}{C'}$$

$$= C_h\ ab_h\ \left[\left(\frac{z}{z_o}\right)^{1/3} - 1\right]^{3/2}$$

Using the value given earlier for a and taking $z \gg z_o$ we have:

$$C_h\ \left(\frac{z}{z_o}\right) = C_h\ b_h\ \frac{k^2}{\left(\ln\frac{z}{z_o}\right)^2}\ \left(\frac{z}{z_o}\right)^{\frac{1}{2}} \qquad (23)$$

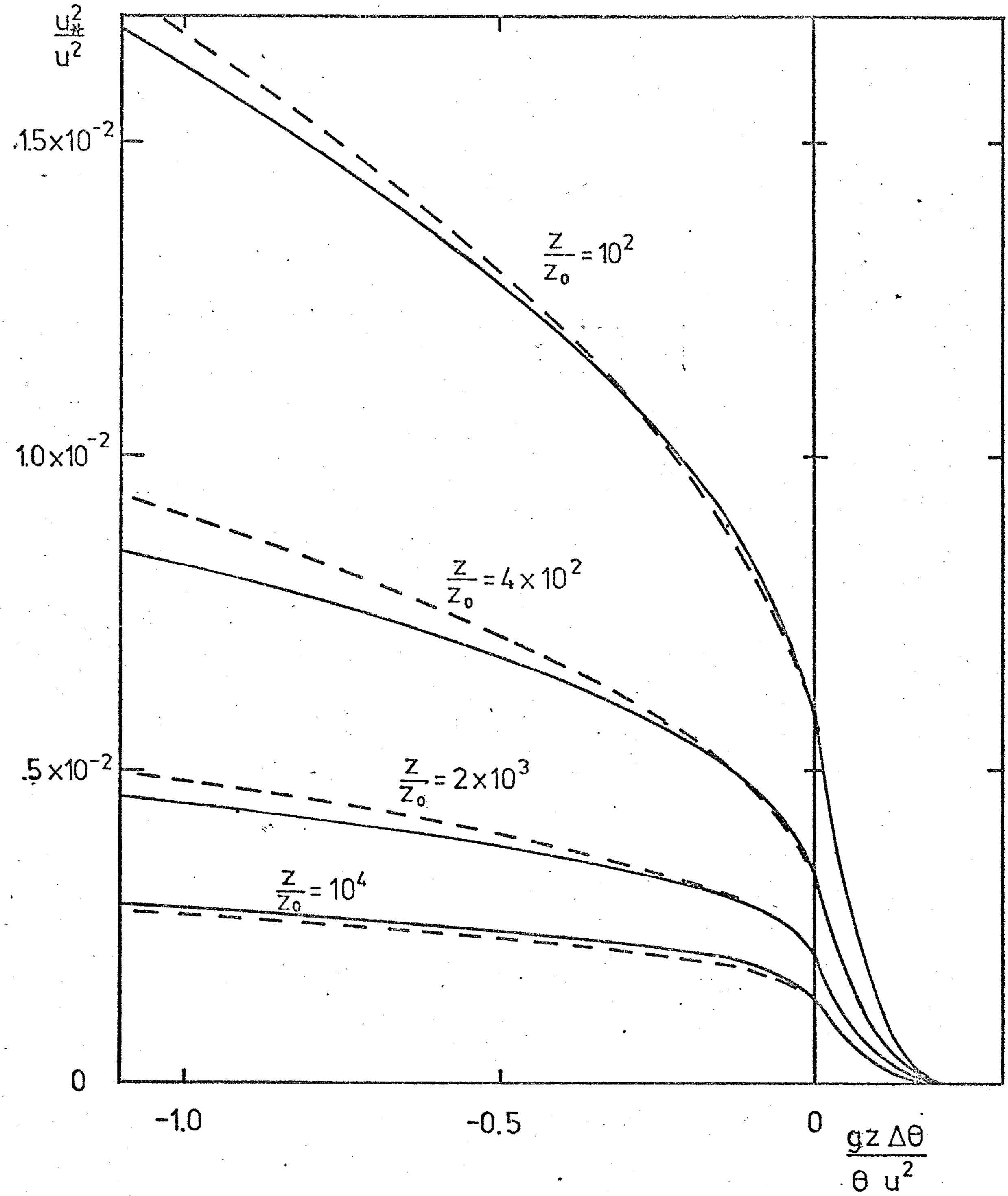

Fig. 3 Drag coefficient for momentum.

________ Similarity theory, iterative computation

-------- Analytical formula.

The curves are undistinguishable for positive stability.

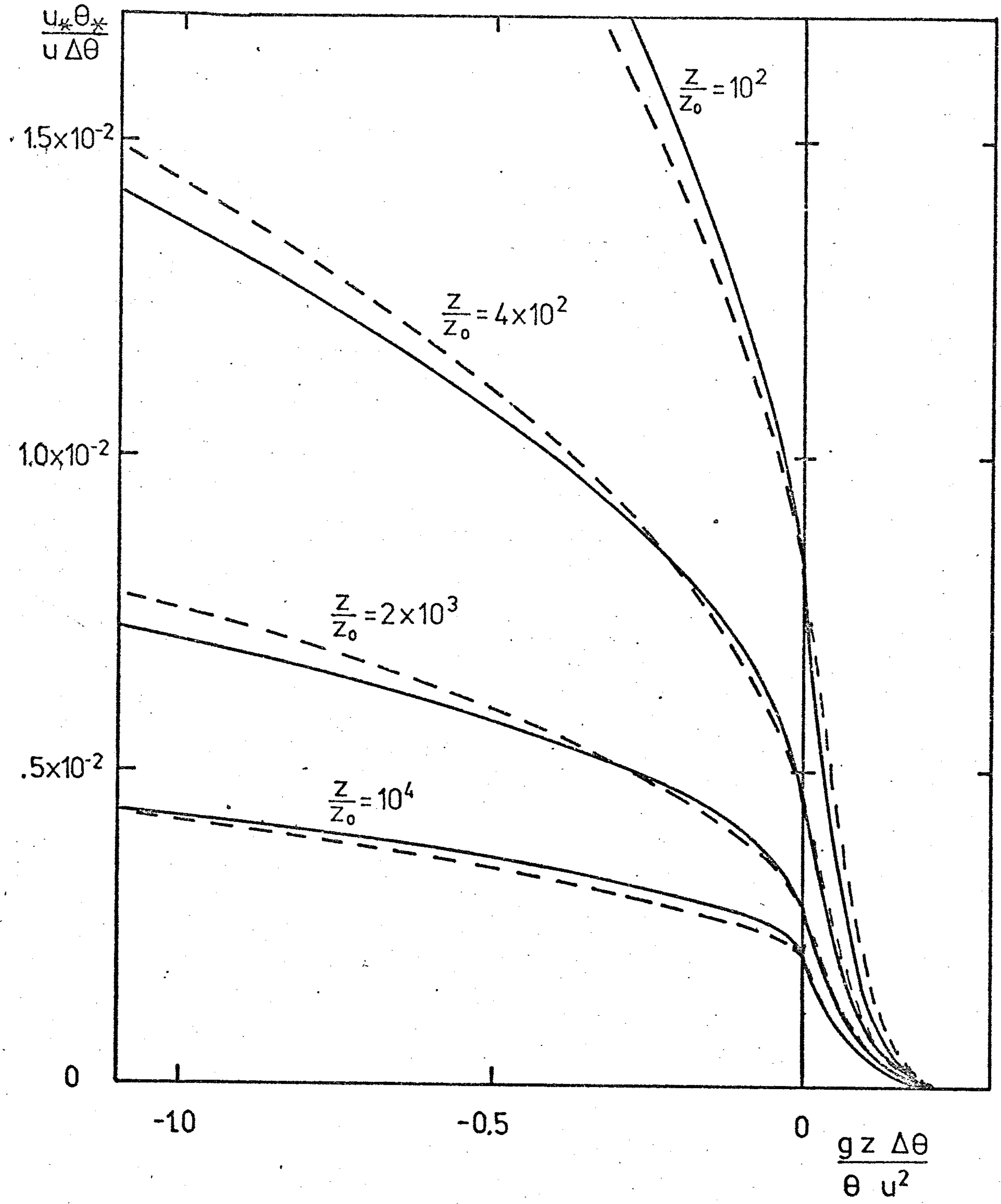

Fig. 4 Drag coefficient for heat.

________ Similarity theory, iterative computation

-------- Analytical formula.

Because of the similarity between the curves in Fig. (3) and (4), and for the sake of symmetry, we suggest a form similar to (18) for the momentum flux in the unstable case :

$$\frac{u_*^2}{u^2} = F(\frac{z}{z_o}, Ri_B) = a\left(1 - \frac{b_m \; Ri_B}{1 + c_m \; a \; b_m (\frac{z}{z_o})^{\frac{1}{2}} \; |Ri_B|^{\frac{1}{2}}}\right) \quad (24)$$

This expression insures that u_*^2 goes to zero for $u \to o$, i.e. for $Ri_B \to -\infty$.

In the stable side we must have F, G and their first derivative equal to zero for $Ri_B = 1/\beta$.

This can be seen by looking at the limit $\zeta \to \infty$ in (14), using (10.b) and (11.b) . Hence if we use second degree polynomials we have

$$F = \frac{u_*^2}{u^2} = a\,(1 - 2\beta \; Ri_B + \beta^2 \; Ri_B^2) \quad (25.a)$$

$$G = \frac{u_* \theta_*}{u \Delta \theta} = \frac{a}{R}\,(1 - 2\beta \; Ri_B + \beta^2 \; Ri_B^2) \quad (25.b)$$

with $Ri_B \leq 1/\beta$ (25.c)

and $a = k^2 / (\ln \frac{z}{z_o})^2$ (25.d)

4.c Further simplification

It can be noticed that the curves suggested for F and G in the stable and unstable regions do not have the same slope at $Ri_B = o$. For the sake of numerical stability in the forecast computation it may be advisable to remove this discontinuity by making $b_m = b_h = 2\beta = 9.4$. The difference between this value and the ones computed from Businger's curves is not great, well within the uncertainty due to the scatter of the observations near $Ri_B = o$. Using this value for b_m and b_h, (18) and (24) can then be used to compute the parameters c_m and c_h for a best fit of the curves. The values that we obtained are $c_m = 7.4$ and $c_h = 5.3$. These curves are shown on Fig. 3 and 4 (dashed lines).

5. Conclusion

In the above discussion it was assumed that the wind was along the x axis. Let us now write the suggested formulae for the surface fluxes of momentum and heat for a general wind direction.

5.a Unstable case

$$\overline{w'u'}(\text{ground}) = a\left[|V| - \frac{b\ x}{|V| + abc_m\left(\frac{z}{z_o}\right)^{\frac{1}{2}}|x|^{\frac{1}{2}}}\right]u \qquad (26.a)$$

$$\overline{w'\theta'}(\text{ground}) = \frac{a}{R}\left[|V| - \frac{b\ x}{|V| + abc_h\left(\frac{z}{z_o}\right)^{\frac{1}{2}}|x|^{\frac{1}{2}}}\right]\Delta\theta \qquad (26.b)$$

with $x = \dfrac{gz\ \Delta\theta}{\theta}$

5.b Stable case

$$\overline{w'u'}(\text{ground}) = a\left(1 - \frac{b\ x}{2|V|^2}\right)^2 |V|\,u \qquad (26.c)$$

$$\overline{w'\theta'}(\text{ground}) = \frac{a}{R}\left(1 - \frac{b\ x}{2|V|^2}\right)^2 |V|\ \Delta\theta \qquad (26.d)$$

for $\dfrac{gz\ \Delta\theta}{\theta\ |V|^2} < \dfrac{2}{b}$

$$\overline{w'u'}(\text{ground}) = \overline{w'\theta'}(\text{ground}) = 0 \qquad (26.e)$$

for $\dfrac{gz\ \Delta\theta}{\theta|V|^2} \geq \dfrac{2}{b}$

The numerical values of the parameters are :

$a = k^2/\left(\ln\frac{z}{z_o}\right)^2$ with $k = 0.35$ (von Karman's constant)

$b = 9.4$, $R = 0.74$, $c_m = 7.4$ and $c_h = 5.3$

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