

RADIATIVE PROCESSES IN THE
ATMOSPHERE

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1. Introduction

The purpose of this course is to develop the theory of radiation transfer in the earth's atmosphere, keeping in mind the particular application of parameterizing radiation processes in numerical forecasting models. Many of the physical processes that must be treated in numerical models are poorly understood, so that parameterizations must necessarily be based on empirical relationships. However, in the case of radiation the basic theory is reasonably well understood, but the direct application to the atmosphere is so complex that gross approximations are essential. To develop these approximations we will go through the following stages:

- . Basic radiation transfer theory,
- . Solutions to simplified problems,
- . Approximations and numerical methods,
- . Parameterization methods.

But first we must be sure that radiation is an important physical process that must be properly treated, so we will begin with a qualitative description of its effects on the atmosphere.

Solar radiation is of course the primary source of energy for all atmospheric processes, and the emission of long wave terrestrial radiation is the final sink of this energy. The atmosphere may be regarded as a heat engine which transfers energy by means of its circulation between the source and sink regions resulting from local imbalances between solar and terrestrial radiation. Thus radiation is fundamental to meteorological processes. A long term climatology will provide an adequate driving term for numerical models, but we must also consider the significance of variations on the short term, and the detailed interactions between such things as the distributions of temperature, humidity, cloud, snow and ice, etc.

The flow of energy per unit area at the mean sun-earth distance is termed the 'solar constant', and has a value of about 1375 Wm^{-2} (Forgan, 1977). Most modern measurements lie within 20 Wm^{-2} of this value. There are variations of about 6 % due to the ellipticity of the earth's orbit, with a maximum in January. Variations of about 1% through the 11 year solar cycle have been reported, but this is far from certain, although the ultraviolet part of the spectrum certainly varies with this period. Due to the geometric factors, the time and space average of incident solar radiation is one quarter of the solar constant, whilst a further 30% is reflected back to space by snow, ice, clouds and the surface. The total mean energy available is about 250 Wm^{-2} . A simple global model will give an effective temperature of the earth's atmospheric system. We may equate the solar input to the emission by a constant temperature black body of the same radius as the earth :

$$\pi R^2 S(1-\alpha) = 4\pi R^2 \sigma T_e^4$$

where πR^2 is the cross sectional area of the earth, and $4\pi R^2$ is its surface area, S is the solar constant, α is the albedo (reflection) and σT_e^4 is the emission from a black body at a temperature T_e . Thus the effective temperature of the earth-atmospheric system is

$$T_e = \left(\frac{S(1-\alpha)}{4\sigma} \right)^{1/4}$$

If we substitute $\alpha=0.3$, $S=1375 \text{ Wm}^{-2}$ and $\sigma=5.67 \times 10^{-8} \text{ Wm}^{-2}\text{K}^{-4}$ we obtain $T_e \sim 255 \text{ K}$, a temperature which is fairly typical of the troposphere. The climatological mean of the surface temperature is about 288 K , somewhat larger than the effective radiating temperature. The difference is a consequence of the so called 'greenhouse effect', which will be discussed later.

We can obtain a general indication of the radiative source and sink regions which drive the atmospheric circulation from satellite measurements of absorbed solar energy and emitted infrared radiation, such as are shown in Fig.1.

There is a surplus of energy of about 60 Wm^{-2} in tropical regions, and a similar deficit in polar regions. The atmosphere and the ocean between them must transport this energy poleward to establish an overall long term balance. We can construct a simple one-dimensional model of this process by assuming that the transport processes can be represented as a diffusion:

$$\sigma T^4(\lambda) - \frac{1}{4} S(\lambda) \{1 - \alpha(\lambda)\} = D \cdot \nabla^2 T(\lambda)$$

where temperature $T(\lambda)$, solar input $S(\lambda)$ and albedo $\alpha(\lambda)$ may depend on latitude. D is a diffusion coefficient which must be chosen to fit. This simple model can give a remarkably realistic fit to the actual temperature distribution, considering its simplicity (North, 1975).

In order to discover whether we need any more complicated treatment of radiation in forecasting models than just climatology, it is necessary to carry out numerical experiments, because there are many complex interrelations with other processes. It seems to be possible to ignore radiation entirely for short range forecasting for perhaps 24 hours, because time scales for radiation are relatively long, and the atmosphere will 'freewheel' under its own inertia. A climatological radiation forcing term will stop it from running down on a longer time scale, but the missing interactions may cause the detailed development to be modelled incorrectly. The most important interactions are probably the direct variations of infrared cooling which are associated with variations in temperature, water vapour and cloud. To a first approximation, temperature perturbations will be eroded by radiative cooling, whilst radiative cooling is much reduced below cloud layers. Further complications are provided by the many

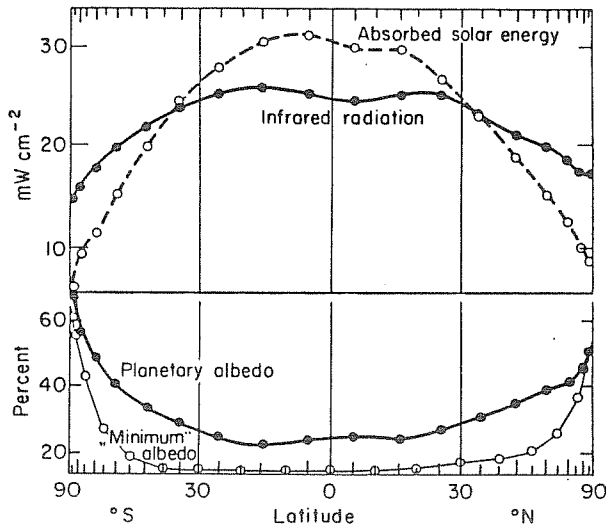


Fig. 1. Mean meridional profiles (averages within latitude zones) of components of the planetary radiation budget measured during the period 1962-1966. The abscissa is scaled by the cosine of latitude. (After Vonder Haar and Suomi, 1971.) Minimum albedo from Nimbus 3 data (T.H. Vonder Haar, private communication, 1975). Paltridge & Platt (1976)

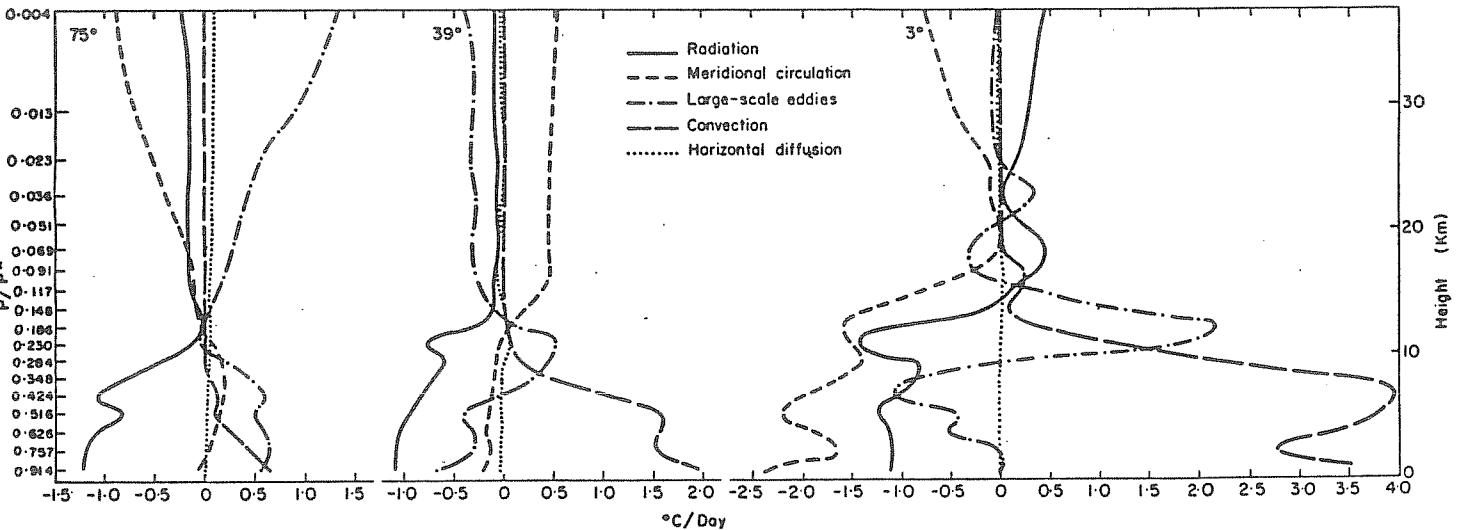


Fig. 2. Components contributing to total cooling rate as a function of height at three latitudes (B.G. Hunt, private communication, 1975). The ordinate is pressure normalized with respect to surface pressure.

Paltridge and Platt (1976)

feedback mechanisms such as the water vapour amplification of the greenhouse effect. The greenhouse effect is an increase in the surface temperature over the radiative equilibrium temperature which would be established in the absence of an atmosphere, because of the increase in downward radiation at the surface due to the atmospheric emission. Water vapour can amplify temperature perturbations on a long time scale, because a higher temperature will increase evaporation, and the extra water will add to the greenhouse effect, thus further increasing the temperature.

As a process for transporting energy vertically, radiation is comparable in importance with the other mechanisms of convection and dynamical transport, as may be seen from Fig.2. It is therefore important for determining the vertical temperature profile. Horizontal transport by radiation is of course negligible, except in the neighbourhood of such inhomogeneities as large cumulus clouds. Fig. 2 gives some indication as to the distribution of radiative heating and cooling, but more information can be obtained from calculations of climatological cross sections such as those of Dopplick (1972). Fig. 3 gives his cross sections for the mean total radiative heating for the northern hemisphere winter period of December to February. We see an overall cooling in most of the troposphere, particularly in low latitudes, with a little heating at low altitudes in the summer hemisphere. The values in the lower stratosphere are somewhat smaller in general, with a heating in low latitudes and a cooling in high latitudes. This diagram is the sum of several different transfer mechanisms, each of which affects different parts of the atmosphere.

Solar radiation is absorbed in the ultraviolet by oxygen in the thermosphere, causing the high temperatures in that region. Ultraviolet and visible radiation are absorbed by ozone in the region of the stratopause, thus causing the relatively high temperatures at around 50 km. Visible and infrared solar radiation is absorbed largely in the troposphere by water vapour and carbon dioxide, and by aerosol, dust and cloud. However the most important absorber of solar radiation is the surface, which then acts as a source of energy, transferring heat into the atmosphere by infrared emission, convection and latent heat. Solar radiation is also scattered by Rayleigh scattering, aerosol, dust and cloud, and is reflected in part by the surface. These mechanisms all combine to determine how much energy is absorbed by the earth/atmosphere system, and where. To be able to calculate the solar heating we need to know the physical mechanisms behind each of the interactions, the geometry of the radiative transfer, the solar spectrum, and the physical state of the atmosphere (i.e. its composition and temperature distribution). Terrestrial radiation is emitted and absorbed by the surface and clouds (both of which may be regarded as nearly black in the infrared, for many practical purposes), and by water vapour, carbon dioxide and ozone. Scattering and reflection is not very important in this part of the spectrum.

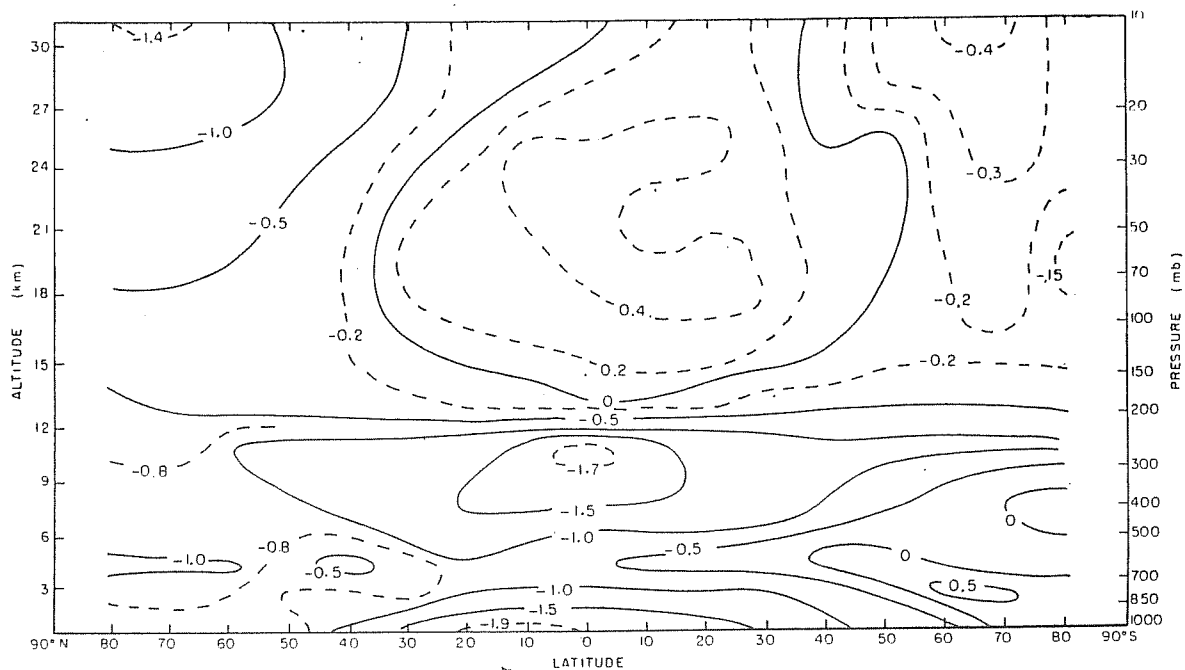


Fig. 3. Mean total radiative heating ($^{\circ}\text{C day}^{-1}$) for December-February.

Dopplick, (1972)

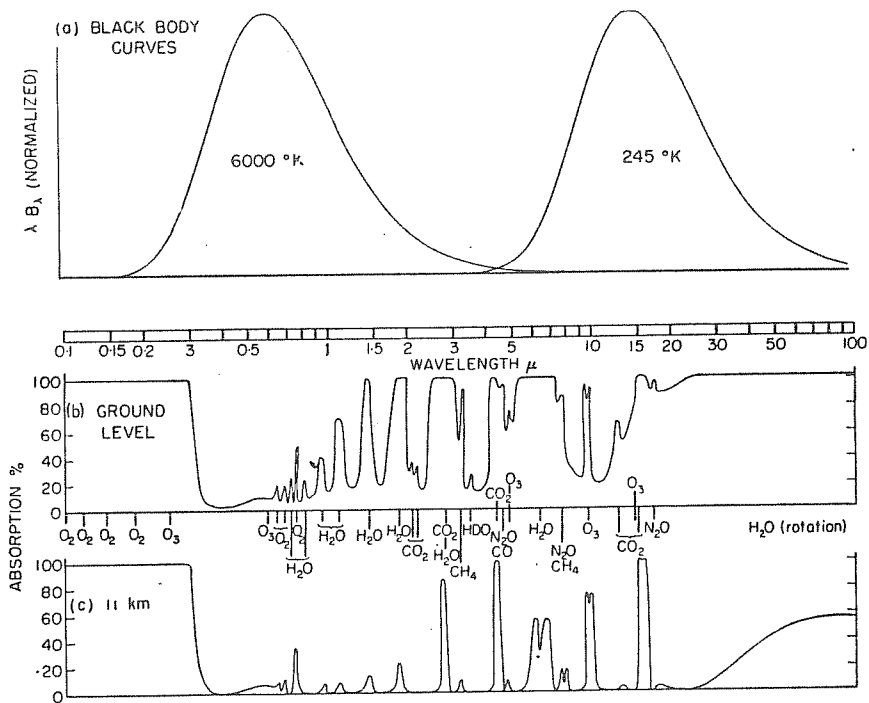


Fig. 4 Atmospheric absorptions.

(a) Black-body curves for 6000°K and 245°K . (b) Atmospheric gaseous absorption spectrum for a solar beam reaching ground level. (c) The same for a beam reaching the temperate tropopause. The axes are chosen so that areas in (a) are proportional to radiant energy. Integrated over the earth's surface and over all solid angles the solar and terrestrial fluxes are equal; consequently, the two black-body curves are drawn with equal areas beneath them. An absorption continuum has been drawn beneath bands in (b). This is partly hypothetical because it is difficult to distinguish from the scattering continuum, particularly in the visible and near infra-red spectrum. Conditions are typical of mid-latitudes and for a solar elevation of 40° or diffuse terrestrial radiation.

Goody, (1964)

Fig. 4 gives an indication of whereabouts in the spectrum the absorption and emission by various gases takes place. The top pair of curves are black body radiation at temperatures typical of the sun and the atmosphere, showing that there is very little overlap spectrally between the two mechanisms. This is of no significance for calculating radiation transfer, as the equations are linear and superposition applies, but is of great help in making measurements. The lower curve is the absorption of the atmosphere along a path from about the tropopause to space, showing which parts of the spectrum and which gases are important for heating and cooling of the stratosphere and above. The middle curve is the absorption along a path from the surface to space. The difference between the lower curves thus indicates regions in which there is absorption or emission in the troposphere. Probably the most obvious feature is the large amount of absorption by water vapour, both in the terrestrial infrared and the solar infrared. We will now return to Dopplick's calculations to show where in the atmosphere transfer by each of the gases is significant. We will only use his diagrams for the winter season, although he also gives results for the other three seasons. Fig. 5 shows solar heating due to ozone. The peak of heating is actually well above the top of the diagram of about 50 km. There is very little in the troposphere, and of course there is none in the polar night. However, there is an indirect effect on the troposphere because ozone will determine in part how much solar radiation is left to be absorbed at the surface. Fig. 6 gives solar heating by the infrared bands of H_2O , CO_2 and O_2 . Here there is very little in the stratosphere, with a maximum in the summer troposphere which is due to a combination of sun angle and day length in the summer, and larger amounts of H_2O in the troposphere. Fig. 7 shows infrared cooling due to H_2O , which is maximum in the upper troposphere, especially in equatorial regions, where temperatures are high. There is less cooling in the lower troposphere even though water vapour density is higher, because of the shielding effect (greenhouse effect) of the strongly absorbing water vapour higher up. The cooling due to CO_2 , as shown in Fig. 8 is completely different. Here the greenhouse effect is even stronger, and the maximum cooling is even higher, at about the stratopause level where it partially balances the ozone solar heating. Cooling is small in the troposphere because photons emitted by CO_2 cannot travel far before they are reabsorbed. The region of heating around the equatorial tropopause is due to that region absorbing radiation from the warmer regions immediately above and below. Infrared transfer in the ozone band (Fig. 9) shows a similar effect, at a somewhat higher altitude, which is largely due to radiation from the surface being absorbed by the lower side of the ozone layer. At higher altitudes there is cooling, as is the case for CO_2 .

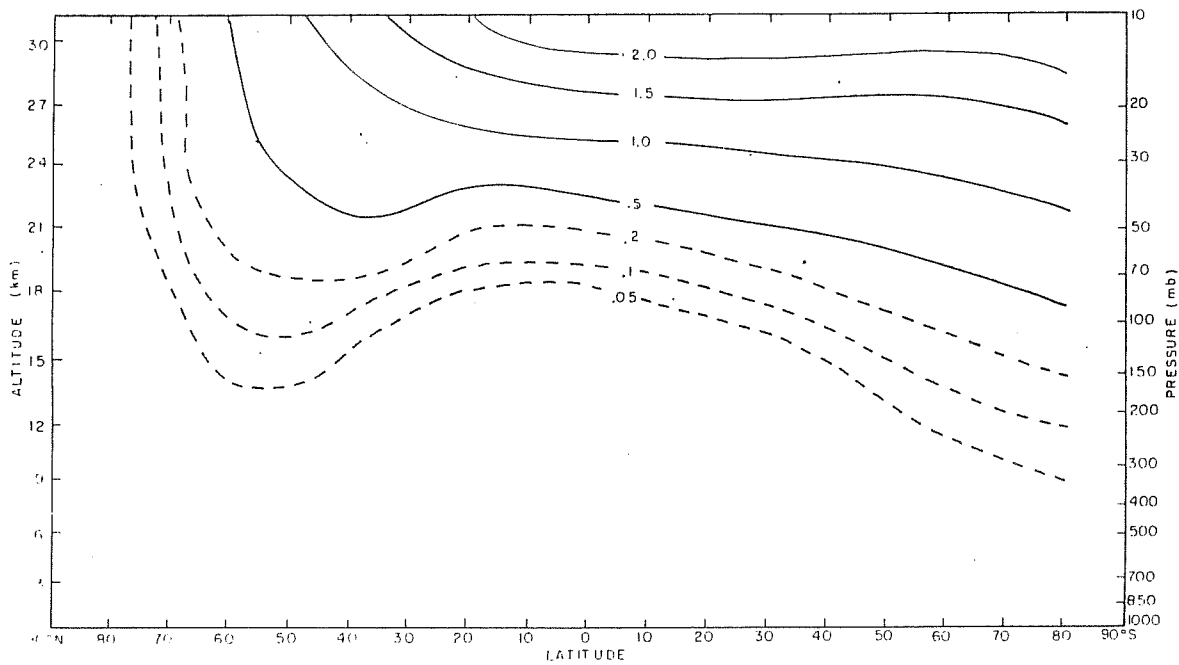


FIG. 5. Mean solar heating ($^{\circ}\text{C day}^{-1}$) by O_3 for December-February.
Dopplick (1972)

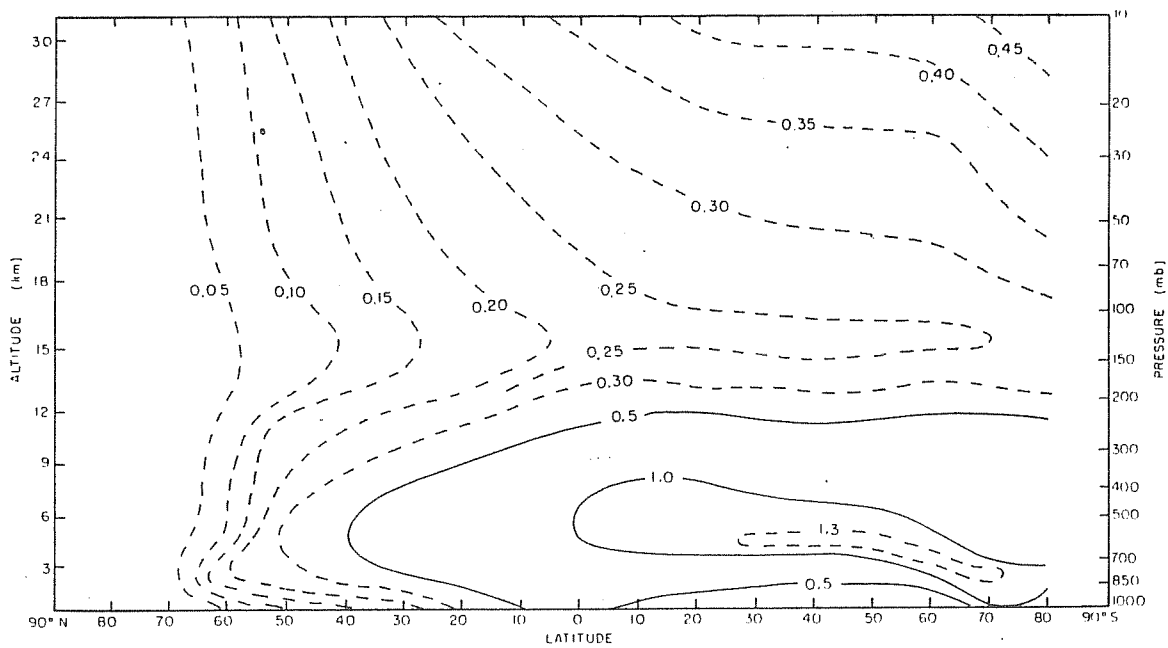


FIG. 6. Mean solar heating ($^{\circ}\text{C day}^{-1}$) by $\text{H}_2\text{O}+\text{CO}_2+\text{O}_2$ for December-February.
Dopplick (1972)

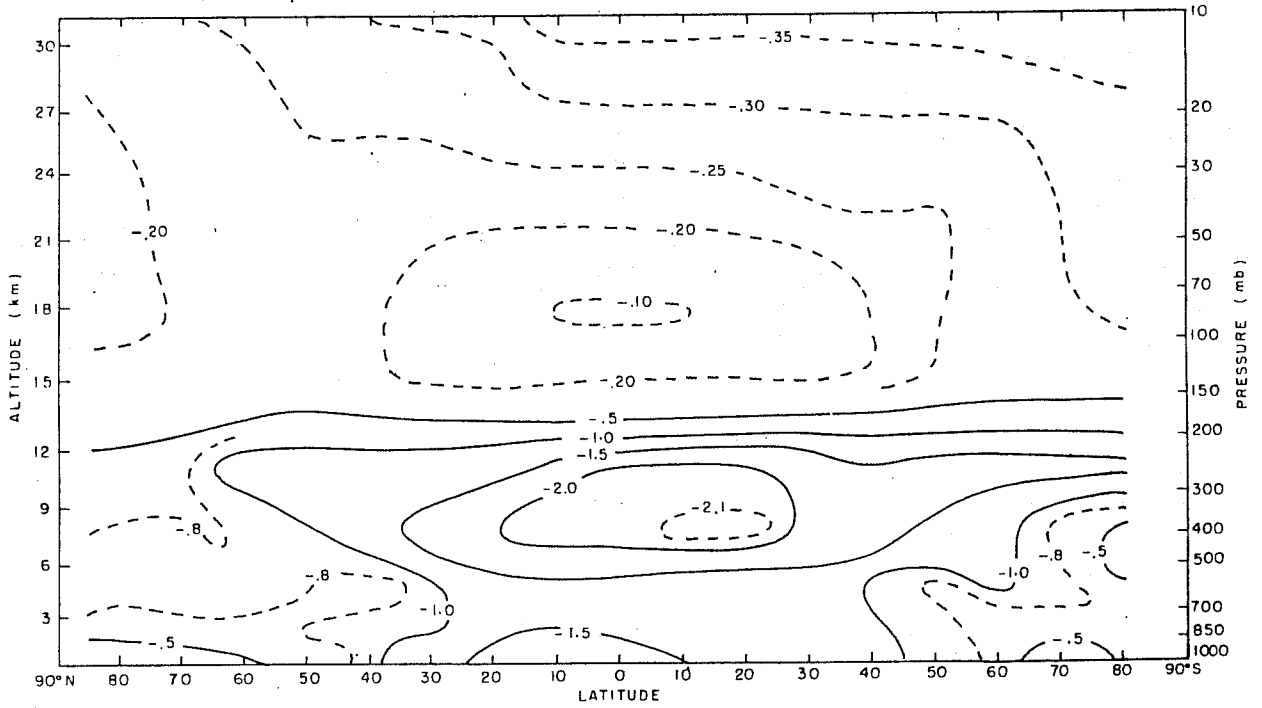


FIG. 7. Mean thermal radiative heating ($^{\circ}\text{C day}^{-1}$) by H_2O for December-February. Dopplick (1972)

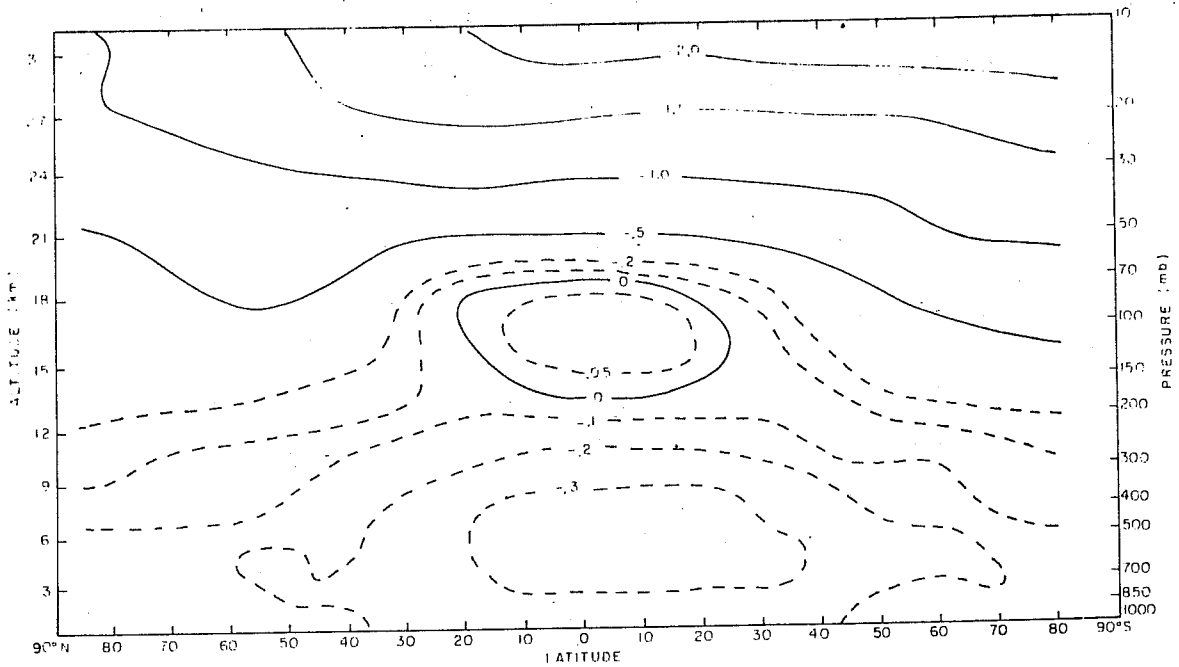


FIG. 8. Mean thermal radiative heating ($^{\circ}\text{C day}^{-1}$) by CO_2 for December-February. Dopplick (1972).

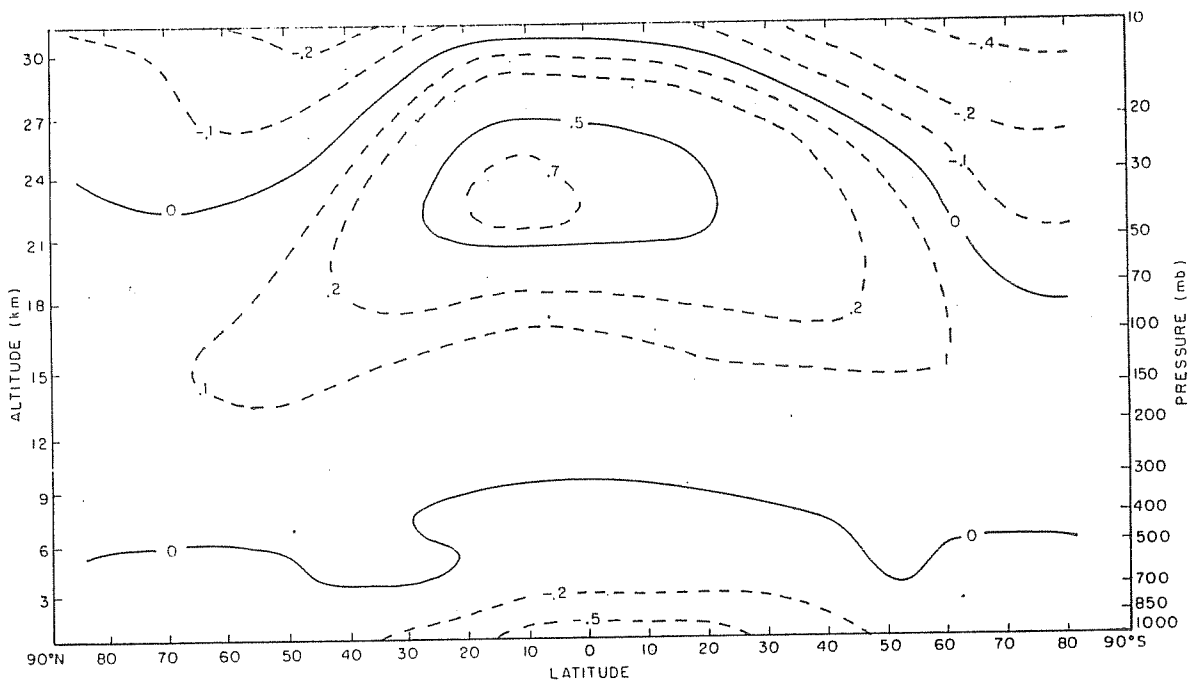


FIG. 9. Mean thermal radiative heating ($^{\circ}\text{C day}^{-1}$) by O_3 for December-February.
Doplick (1972)

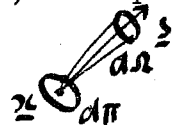
2. Radiative Transfer Theory

In this section we will derive the equation of transfer in semi-formal manner, and find some solutions of simplified problems to illustrate how the equation may be used. We will start with some basic definitions.

The wavelength of radiation is given the symbol λ , and is usually measured in nanometres or micrometres. The frequency may be measured in Hertz, or may be by angular frequency, ω , measured in radians s^{-1} , but a much more commonly used quantity is the wave number ν , (the spatial frequency) which is generally measured in cm^{-1} .

The specific intensity I is the rate of flow of radiant energy per unit solid angle per unit frequency, wave number, or wave length interval. For example, referred to wave number, the quantity

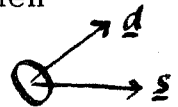
$$I(\underline{x}, \underline{s}, \nu, t) d\pi d\Omega d\nu dt$$



is the energy crossing an area $d\pi$ located at position \underline{x} into a solid angle $d\Omega$ about the direction \underline{s} in wave number interval $d\nu$ in time dt . Radiation is said to be isotropic if I is independent of direction \underline{s} , and homogeneous if it independent of position \underline{x} . We will normally not consider variations of radiation with time t .

The flux density F is an integral of specific intensity over angle. It is the rate of flow of energy per unit area per unit wave number (frequency or wave length) regardless of direction. If F_d is the flux across an area normal to direction \underline{d} , then

$$F_d(\underline{x}, \nu, t) = \int I(\underline{x}, \underline{s}, \nu, t) \underline{d} \cdot \underline{s} d\Omega$$



It can be shown (Goody 1964) that we can define a vector flux \underline{F} with three components at right angles for example :

$$\underline{F} = (F_x, F_y, F_z)$$

The energy density $u(\underline{x}, \nu)$ of a radiation field is simply the radiant energy per unit volume per unit wave number. It may be derived from the specific intensity as follows :

The total energy within a volume $d\pi dy$ in a solid angle $d\Omega$



is the energy crossing $d\pi$ in time $dt = dy/c$:

$$E = I(\underline{x}, \underline{s}, \nu) d\pi \frac{dy}{c}$$

The density of radiation flowing in the direction \underline{s} in $d\Omega$ is therefore $E/d\pi dy$, i.e. $\frac{1}{c} I(\underline{x}, \underline{s}, \nu)$. Therefore the total energy density is

$$u(\underline{x}, \nu) = \int \frac{1}{c} I(\underline{x}, \underline{s}, \nu) d\Omega$$

$$= \frac{4\pi}{c} \bar{I}(\underline{x}, \nu)$$

where we define the average intensity \bar{I} as $\int I d\Omega / \int d\Omega$.

The heating function $h(\underline{x}, \nu)$ is the rate at which a unit volume gains energy. It may be regarded as the time derivation of energy density

$$h(\underline{x}, \nu) = \frac{\partial}{\partial t} u(\underline{x}, \nu, t) = \frac{4\pi}{c} \frac{\partial}{\partial t} \bar{I}(\underline{x}, \nu, t)$$

but is more usefully found from the spatial derivative of the vector flux:

$$h(\underline{x}, \nu) = -\nabla \cdot \underline{F}(\underline{x}, \nu)$$

It is left to the reader to prove that these two definitions are in fact equivalent.

The specific intensity, flux density, energy density and heating function have all been defined spectrally. There are corresponding definitions for spectrally integrated quantities, and the two definitions should normally be distinguished by means of the suffix ν and the word 'spectral', e.g. spectral flux density, F_ν , but as we will usually be dealing with spectral quantities or properties that apply to both spectral quantities and integrated quantities, the prefix will be omitted for conciseness.

The Equation of Transfer

All of the above quantities are purely geometric properties of the radiation field alone. We will now consider interactions between radiation and matter.

Radiation may be absorbed, scattered, or emitted by matter. The amount of radiation removed from an incident beam by absorption and scattering is proportional to the intensity and also to the amount of matter, according to Beer's Law. The constant of proportionality is called the extinction coefficient,



$$dI = -eI dx$$

The extinction coefficient may be defined in terms of distance, as here, or absorber mass, or number of molecules, according to the requirements of the problem in hand. The product $e \cdot dx$ is dimensionless and is independent of the coordinate system. Its integral is known as the optical depth τ , defined by

$$d\tau = e dx$$



so that

$$dI_1 = -I d\tau$$

The amount of radiation added to the beam by emission and scattering may be written in the same form by defining the source function J :

$$dI_2 = J d\tau$$

The source function will be described in more detail later; in the particular case of no scattering and thermodynamic equilibrium, it becomes the Plank or black body function. Combining the two terms, we obtain the equation of transfer in its most basic form

$$dI = dI_1 + dI_2 = -I d\tau + J d\tau$$

i.e.

$$\frac{dI}{d\tau} = J - I$$

The equation may be integrated explicitly by means of an integrating factor e^τ :

$$\frac{dI(\tau)e^\tau}{d\tau} + I(\tau)e^\tau = J(\tau)e^\tau$$

Integration between the ends of a part from 0 to τ_1 gives:

$$[I(\tau)e^\tau]_0^{\tau_1} = \int_0^{\tau_1} J(\tau)e^\tau d\tau$$

i.e.

$$I(\tau_1)e^{\tau_1} - I(0) = \int_0^{\tau_1} J(\tau)e^\tau d\tau$$

rearranged to give an expression for I (τ_1) :

$$I(\tau_1) = I(0)e^{-\tau_1} + \int_0^{\tau_1} J(\tau)e^{-(\tau_1-\tau)} d\tau$$

Each term in this equation has a simple physical interpretation. The first term is the intensity I (0) incident at $\tau = 0$ multiplied by the transmission of the path $e^{-\tau_1}$.

The element J (τ) $d\tau$ of the integral is the emission of the element of path $d\tau$, and $\exp(-(\tau_1-\tau))$ is the transmission of the path from τ to τ_1 . Thus we may define transmission T (τ) as

$$T(\tau) = e^{-\tau}$$

The Source Function

The source function is defined as the emission per unit optical depth, and comprises two mechanisms - thermal emission and scattering. We will only consider here the situation known as 'local thermodynamic equilibrium' (LTE) which is applicable throughout the troposphere and lower stratosphere for all significant atmospheric absorption bands. LTE holds when the excited states of absorbing molecules are populated predominantly by collisional processes rather than by radiation. The emission component of the source function is then given by $B d\tau_a$, where B is the Planck function, and τ_a is the optical depth for absorption (or emission).

The scattering component is more complicated, as it involves scattering of intensity I (\underline{d}) from every incoming direction (\underline{d}) into the outgoing direction \underline{s} under consideration; according to the 'Phase function' P ($\underline{d}, \underline{s}$) which describes the efficiency of the scattering from \underline{d} to \underline{s} . The total emission of an element $d\tau$ may be written

$$J(\underline{s})d\tau = B d\tau_a + d\tau_s \int_{4\pi} I(\underline{d}) P(\underline{d}, \underline{s}) d\Omega_d$$

where $d\tau_s$ is the optical depth for scattering, $d\tau = d\tau_a + d\tau_s$ and P($\underline{d}, \underline{s}$) is normalised so that its integral over all angles is unity. The integral over solid angle Ω_d refers to all directions \underline{d} . If we denote the absorption coefficient by a, and the scattering coefficient by s, we may define the "albedo of single scattering ω " as the fraction of incident radiation which is scattered out of an element of optical path:

$$\omega = \frac{d\tau_s}{d\tau} = \frac{d\tau_s}{d\tau_a + d\tau_s} = \frac{s}{s+a}$$

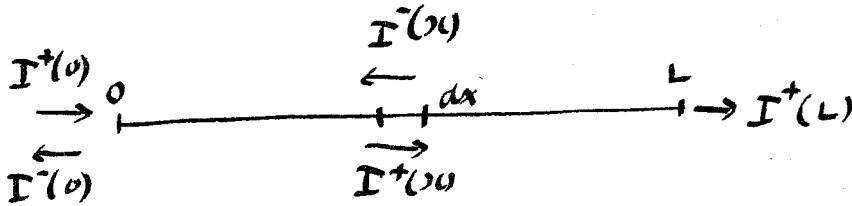
and the expression for the source function becomes

$$J(\underline{s}) = (1-\omega)B + \omega \int I(\underline{d}) P(\underline{d}, \underline{s}) d\Omega_d$$

The two stream approximation

We may use the two stream approximation i.e. one-dimensional radiative transfer as a simple example of the use of the radiative transfer in order to obtain an idea as to how solutions are likely to behave. In the full 3-dimensional case, it is rare to obtain explicit algebraic solutions, and normally numerical techniques must be used.

Let us consider first a one-dimensional absorbing and scattering medium, with radiation I(o) incident at one end and no significant thermal emission. This could, for example, be a model for transfer of sunlight vertically through a plane parallel cloud



We will use distance as the coordinate, rather than optical depth, so that we can use the absorption and scattering coefficients a and s explicitly. Let the path be of length L , and have radiation $I^+(0)$ incident at $x = 0$. The intensities in the two streams will be denoted by $I^+(x)$ and $I^-(x)$ respectively and will be regarded as positive quantities. The equation of transfer for each stream is

$$\frac{dI^+}{dx} = -(s+a)I^+ + \frac{s}{2}(I^+ + I^-)$$

$$\frac{dI^-}{dx} = +(s+a)I^- - \frac{s}{2}(I^+ + I^-)$$

The first term on the right-hand side is due to extinction, and the second term is the source term due to scattering. The phase function chosen is one half each way. This is a pair of linear simultaneous differential equations that can easily be solved explicitly. The complete solution is left as an exercise for the reader; only two particular cases are considered here, namely no scattering, $s = 0$, and no absorption, $a = 0$.

In the no scattering case, the solution is trivial, and as expected:

$$I^+(x) = I^+(0)e^{-ax} ; \quad I^-(0) = 0$$

giving an exponential transmission e^{-ax} or $e^{-\tau}$ multiplied by the incident intensity $I^+(0)$.

$I^-(0)$ is zero because there is no incident radiation at $x=L$ in the case chosen for this illustration. In the case of no absorption, $a = 0$, the equations are still coupled:

$$\frac{dI^+}{dx} = \frac{s}{2}(I^- - I^+)$$

$$\frac{dI^-}{dx} = \frac{s}{2}(I^- - I^+)$$

if we take the difference between the equations we find

$$\frac{d}{dx}(I^+ - I^-) = 0$$

therefore

$$I^+(x) - I^-(x) = \text{constant} = I^+(L)$$

substitute back in the first equation

$$\frac{dI^+}{dx} = \frac{s}{2} I^+(L)$$

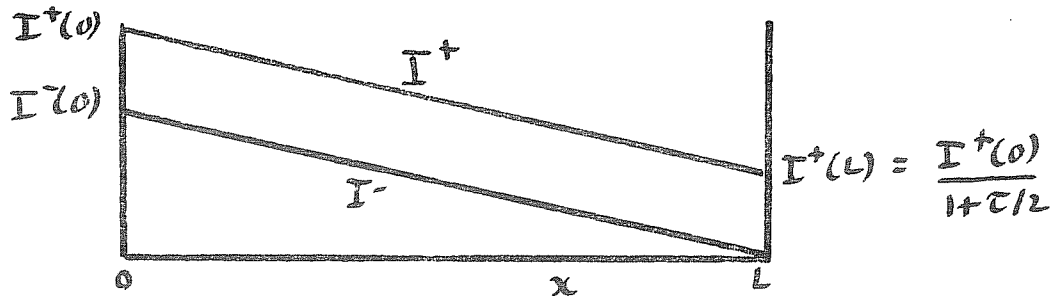
On integrating between 0 and L

$$I^+(L) - I^+(0) = -\frac{sL}{2} I^+(L)$$

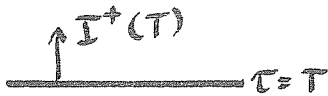
so that

$$I^+(L) = I^+(0) / (1 + sL/2)$$

Thus the transmission of a purely scattering path is $2/(2+sL)$ which contrasts sharply with e^{-sL} for absorption. Integrating between 0 and x shows that $I^+(x)$ is linear in x, as the difference $I^+ - I^-$ is constant, so must $I^-(x)$ also be linear:



A similar problem that we can apply two stream theory to is that of a radiative equilibrium atmosphere. Here we will deal with spectrally integrated quantities, so we assume that the absorption coefficient is a constant, independent of position and wavelength.



Consider a black surface maintained at a temperature θ_g , so that its emission is

$$B_g = \sigma \theta_g^4$$



and an atmosphere of total optical depth T. Ignore other processes such as molecular conduction and convection.

The equations of transfer for the upward and downward streams are



$$\frac{dI^+}{d\tau} = B(\tau) - I^+(\tau)$$

$$\frac{dI^-}{d\tau} = -B(\tau) + I^-(\tau)$$

where the source function has been replaced by $B = \sigma \theta^4$. These equations appear to be uncoupled, but in fact they are linked through the condition for equilibrium, that the heating should be zero

$$h = \frac{d(I^+ - I^-)}{d\tau} = 0$$

In two stream theory, there is no distinction between intensity and flux. Integrate this equation, and we find that

$$I^+(\tau) - I^-(\tau) = \text{constant} = I^+(\tau)$$

so that the difference between the two streams is constant.

If we substitute from the equations of transfer into the equilibrium condition we obtain

$$B(\tau) = \{I^+(\tau) + I^-(\tau)\}/2$$

so that the $B(\tau)$ is the average of the two streams. At the surface, $I^+(0) = B_g$, so that in general $B(0) \neq B_g$, and there is a difference between the atmosphere temperature at the surface and the surface temperature. In terms of the black body emission this difference is

$$B_g - B(0) = \frac{1}{2} I^+(\tau)$$

as may easily be verified by eliminating I^- between the previous two equations evaluated at $\tau = 0$. In a real atmosphere this discontinuity would be rapidly eroded by conduction and convection. At the top of the atmosphere $I^-(T) = 0$, so that

$$B(\tau) = \frac{1}{2} I^+(\tau)$$

The size of this discontinuity may be derived in terms of B_g and T by adding the equations of transfer

$$\frac{d}{d\tau} (I^+(\tau) + I^-(\tau)) = I^-(\tau) - I^+(\tau)$$

therefore

$$2 \frac{dB(\tau)}{d\tau} = I^+(\tau)$$

so that, an integrating between 0 and T

$$B(\tau) - B(0) = \frac{1}{2} I^+(\tau) \cdot \tau$$

substitute for $B(T)$ and $B(0)$, and we obtain

$$I^+(\tau) = \frac{2 B_g}{2 + T}$$

which is identical to the expression for transfer through a pure scatterign layer. The context is different, but the mechanism has the same effect - the energy leaving an element of optical depth must be the same as the energy entering in both cases. This equation may also be regarded as a description of the greenhouse effect. If the surface temperature is maintained by the downward thermal emission $I^-(0)$ and solar radiation S which passes unattenuated through the atmosphere in the visible region, then equilibrium requires that

$$S = I^+(T)$$

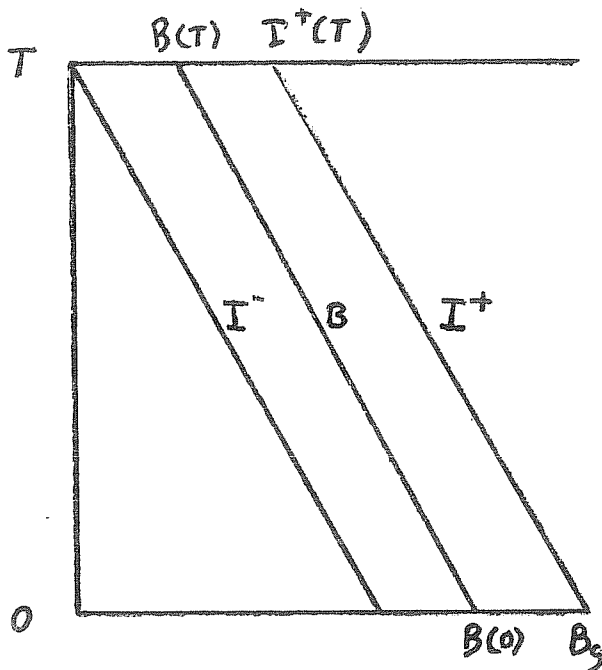
and the surface temperature is given by θ_s , where

$$\sigma \theta_s^4 = B_s = \left(1 + \frac{T}{2}\right) S$$

so that an atmosphere of total optical depth T raises the surface temperature by a factor $(1 + T/2)^{1/4}$ above the value in the absence of an atmosphere. In the case of the earth, an optical depth of 1.25 would be required to raise the temperature from 255 K to the observed climate average of 288 K. Of course this is only an order of magnitude calculation, because the earth's atmosphere does not have a constant absorption coefficient, and it does have convection. The profile of $B(\tau)$ is easily shown to be

$$B(\tau) = B(0) - \frac{I^+}{2} \tau$$

which is linear in optical depth. I^+ and I^- must therefore also be linear in optical depth.



Equation of Transfer for Flux

For practical purposes we are particularly interested in the overall vertical transport of energy in a plane parallel atmosphere, i.e. the wave length or wave number integrated flux density. To evaluate this quantity we need to integrate intensity over both angle and wave number. In the case of symmetry about the vertical, we may note that the vector flux is directed vertically, so that horizontal components are not needed. We will treat the upward and downward parts of the flux density separately for convenience.

$$F^{\uparrow}(z) = \int_0^{\infty} F_{\nu}^{\uparrow}(\nu, z) d\nu = \int_0^{\infty} \int_0^{2\pi} \int_0^{\pi/2} I_{\nu}(\nu, z, \theta, \phi) \cos\theta \sin\theta d\theta d\phi$$

The symmetry implies that I_{ν} is independent of ϕ , so that $\int d\phi = 2\pi$, and the averaging operator becomes

$$2\pi \int_0^{\infty} d\nu \int_0^{\pi/2} \cos\theta \sin\theta d\theta$$

This average can be applied to the whole equation of transfer, because it is linear in I . Because the absorption coefficient is likely to vary with wave length, we must use distance, z , rather than optical depth τ , as a coordinate, and it is also convenient to use a mass absorption coefficient, so that absorber density $\rho(z)$ is also required:

$$d\tau = k(z)\rho(z)dz \quad \tau(z) = \int_0^z k(z')\rho(z')dz'$$

The integral form of the equation of transfer is now

$$I(z) = I(0) \exp\left(-\int_0^z k(z') \rho(z') dz'\right) + \int_0^z B(z') \exp\left\{-\int_{z'}^z k(z'') \rho(z'') dz''\right\} k(z') \rho(z') dz'$$

where we have assumed that there is no scattering, and local thermodynamic equilibrium, so that the source function $J = B$, the Planck function. The integral can be simplified to read:

$$\int_0^z B(z') \frac{dT(z, z')}{dz'} dz'$$

Or, in terms of transmittance $T(z, z')$:

$$I(z) = I(0) T(0, z) + \int_0^z B(z') \frac{dT(z, z')}{dz'} dz'$$

We may apply the angle averaging operator to this equation to give the transfer equation for flux density :

$$F_V^\uparrow(\nu, z) = 2\pi \int_0^{\pi/2} I(0, \theta) \exp\left\{-\int_0^z k(z') \rho(z') \sec\theta dz'\right\} \sin\theta \cos\theta d\theta + 2\pi \int_0^{\pi/2} \int_0^z B(z') \frac{dT(z, z')}{dz'} \exp\left\{-\int_{z'}^z k(z'') \rho(z'') \sec\theta dz''\right\} \sin\theta \cos\theta d\theta dz'$$

where the $\sec\theta$ appears in the optical depth integral because the transmittance is along a slant path at angle θ to the vertical. We may define a FLUX TRANSMITTANCE T_f :

$$T_f(z, z') = 2 \int_0^{\pi/2} \exp\left\{-\int_{z'}^z k(z'') \rho(z'') \sec\theta dz''\right\} \sin\theta \cos\theta d\theta$$

and assume that $I(0, \theta)$ is independent of θ , as it will be if the surface is a black body, then

$$F_V^\uparrow(\nu, z) = F_V^\uparrow(\nu, \theta) T_f(0, z) + \int_0^z \pi B(z') \frac{dT_f(z, z')}{dz'} dz'$$

which is of the same form as the equation for intensity. The equation for the downward flux is almost identical, except for the limits of the integration and the boundary term, which may be zero or the emission from clouds.

All that remains is the wavenumber integration, which is straightforward in principle, if lengthy in practise.

3. Transmission of Radiation by Atmospheric Gases

Before we can discuss approximations for calculating the transmittance of atmospheric gases, we must discuss briefly the physical mechanisms involved in the absorption and emission of photons by molecules.

3.1 Molecular Spectra

A spectral line corresponds to a change of the internal energy of a molecule. This energy is made up of four components, electronic E_e , vibrational E_v , rotational E_r and translational E_t .

$$E = E_e + E_v + E_r + E_t$$

Of these, the first three are quantised, which means that they can only take one of several discrete values. A consequence is that any change ΔE in internal energy can only take discrete values, so that only one particular wavelength of radiation can be absorbed:

$$h\nu = \Delta E$$

Because of this simple relationship between energy and wave-number, it is often convenient to speak of energies in terms of the corresponding wavenumber. Thus typical changes of rotational energy correspond to $\Delta E_r \sim 0-500 \text{ cm}^{-1}$, giving spectral lines in the far infrared and microwave, changes in vibrational energy typically are $\Delta E_v \sim 500-2000 \text{ cm}^{-1}$, giving lines in the infrared, and typical changes of electronic energy are $\Delta E_e \sim 10,000 \text{ cm}^{-1}$, giving lines in the visible and ultraviolet. All three components may change simultaneously, so that a vibrational transition will have rotational structure, giving rise to a vibrational-rotation band of lines, and an electronic transition may have both vibrational and rotational structure. It is vibration-rotation bands which are the main feature of the infrared spectrum, and they are dominant in transferring terrestrial radiation.

Let us consider as an example the rotation of a simple linear molecule such as carbon dioxide. As it is linear, it only has one rotation, about an axis at right angles to the molecule. Its angular momentum $I\omega$ can only take one of a set of discrete values according to its rotational quantum number J :

$$I\omega = \frac{h}{2\pi} \sqrt{J(J+1)}$$

The energy of rotation is

$$E_r = \frac{1}{2} I \omega^2 = \frac{h^2 J(J+1)}{8\pi^2 I}$$

Only changes of ± 1 or zero in rotational quantum number J are allowed, so that for $\Delta J=1$, $\Delta E=2BJ$, where $B = h^2/8\pi^2 I$. Thus lines are spaced at intervals of $2B$ in this simple case. There are some complications which cause the spacing to be slightly variable, such as centrifugal stretching of the molecule, which causes the moment of inertia I to depend on rotation, and this adds a term $-DJ^2 (J+1)^2$ to the energy. Also, B may depend on the vibrational state, so that a typical change of energy is

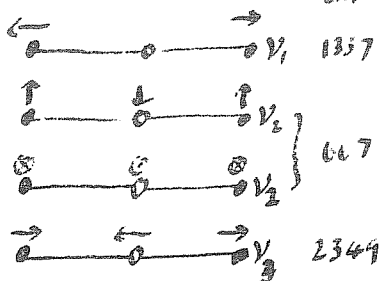
$$\Delta E = (B' + B'')J + (B' - B'')J^2 - 2DJ^3$$

giving a line spacing of

$$\Delta \nu = B' + B'' + 2(B' - B'')J - 6DJ^2$$

Fig. 10 shows a spectrum of the CO_2 bands near 2.7μ , illustrating the nearly constant line spacing.

The vibration of carbon dioxide can take place at one of three fundamental frequencies, together with harmonics, and at ω_i combinations of these frequencies.



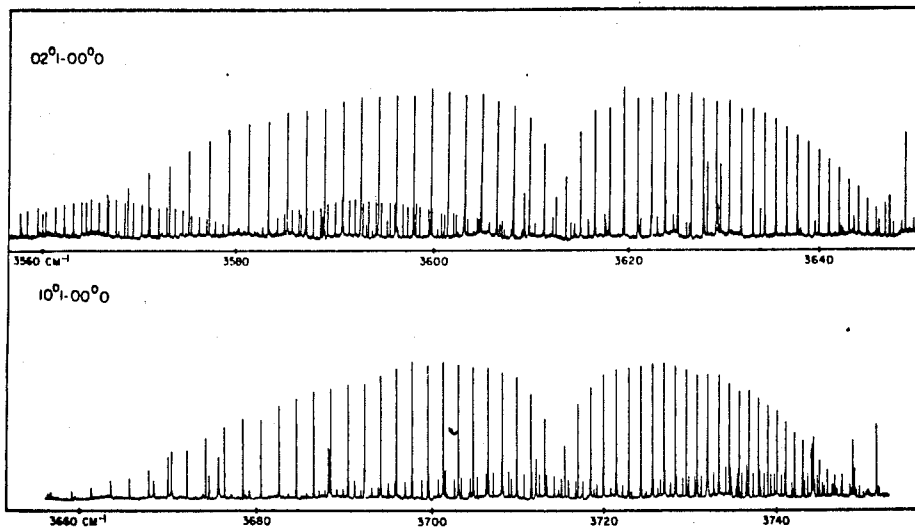
The modes are illustrated in the diagram.

The molecule can have a total energy which is made up of an integral number of quanta in each mode :

$$E_v = (v_1 + \frac{1}{2})h\nu_1 + (v_2 + \frac{1}{2})h\nu_2 + (v_3 + \frac{1}{2})h\nu_3$$

where v_i are the vibrational quantum numbers, ν_i are the vibrational frequencies, and the $\frac{1}{2}$ is the "zero point energy" which is a consequence of the uncertainty principle.

The 15μ band corresponds to a change of one in v_2 , and the 4.3μ band is a change of one in v_3 . There is no band at 1337 cm^{-1} , corresponding to ν_1 , as the symmetric stretching vibration has no dipole moment and cannot interact with radiation. The water molecule is non linear, so that it has three possible rotation axes. The result is that the spectrum is much more complicated, and in the example of Fig. 11 the spectral lines appear to be placed almost at random. This apparent randomness is an important feature when developing models for the transmittance of water vapour as we will see later.



Rapid scans of the $02^{\circ}1-00^{\circ}0$ and $10^{\circ}1-00^{\circ}0$ bands and their associated $\Pi-\Pi$ hot-bands. The respective path lengths were 10.5 and 5.5 cm and the pressure was 14 torr.

Fig. 10 CO_2

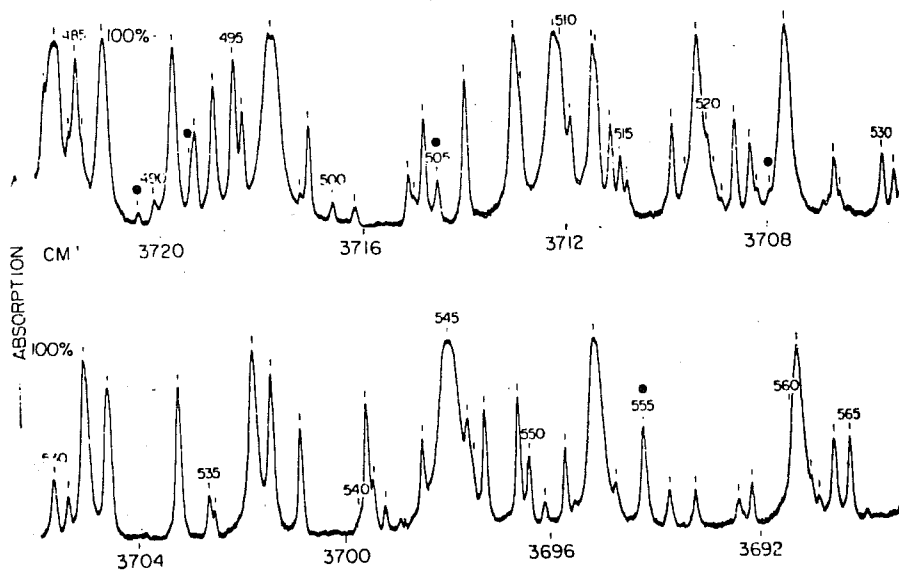


Fig. 11 H_2O

The shape of spectral lines

A spectral line is not infinitely narrow, a single frequency at which absorption takes place, because there are several broadening mechanisms which make it into a finite region with an absorption coefficient which depends on wavenumber. The three most important mechanisms are natural broadening, which is a consequence of the uncertainty principle; Doppler broadening, which is due to the motion of molecules; and collision broadening, which is due to the disturbance of the internal energy levels of a molecule during a collision. Of these, only the last two are significant for atmospheric processes.

The Doppler shift of a spectral line nominally at wavenumber ν_0 due to a line of sight velocity of u is

$$\nu - \nu_0 = \frac{u}{c} \nu_0$$

where c is the velocity of light. The distribution of velocities is Maxwellian

$$P(u) = \exp(-mu^2/2kT)$$

where m is the mass of the molecule. The spectral line shape is the distribution of Doppler shifts, and is proportional to

$$\exp(-m^2c^2(\nu - \nu_0)^2/2kT\nu_0^2)$$

If this is normalised, we can write it as a Doppler line shape factor $f_D(\nu)$

$$f_D(\nu) = \frac{1}{\alpha_D \sqrt{\pi}} \exp\left\{-\frac{(\nu - \nu_0)^2}{\alpha_D^2}\right\}$$

where the Doppler width parameter $\alpha_D = \frac{\nu_0}{c} \left(\frac{2kT}{m}\right)^{1/2}$. Typical value of α_D for infrared bands of atmospheric molecules are around 10^{-3}cm^{-1} .

Collision broadening is a very complex subject if treated properly according to quantum mechanics, but fortunately a very simple classical approach will yield a fairly accurate approximation to the line shape. Assume that the molecule is a classical oscillator, emitting a continuous wave form at frequency ν_0 of constant amplitude, which changes randomly in phase at every collision.



We may compute the spectrum of this wave form by averaging the spectrum of a single segment over random distribution of segment lengths. The spectrum of a segment of length t is the real part of

$$S(\nu, t) = \frac{1}{\pi} \int_0^t e^{i\nu_0 t} e^{-i\nu t'} dt'$$

The distribution of segment lengths is Poisson, the probability density function being

$$P(\tau) = \frac{1}{\tau} e^{-\tau/\tau}$$

where τ is the mean length, or mean time between collisions. The line shape is therefore the real part of

$$L = \int_0^{\infty} P(\tau) S(\nu, \tau) d\tau = \frac{1}{\tau} \int_0^{\infty} d\tau \int_0^{\tau} d\tau' e^{-\tau'/\tau} e^{-i(\nu-\nu_0)\tau'}$$

The evaluation of the real part of this integral is straightforward, and is left as an exercise to the reader. The resulting collision broadened line shape factor is the Lorentz profile

$$f_L(\nu) = \frac{1}{\pi} \frac{1/\tau}{(\nu-\nu_0)^2 + (1/\tau)^2}$$

The half width $\alpha_L = 1/\tau$ in terms of frequency or $1/c\tau$ in terms of wave number. If molecules are regarded as billiard balls, classical kinetic theory gives

$$\alpha_L = \frac{nV}{2\pi c} \sigma = \alpha_0 \frac{P}{P_0} \left(\frac{T_0}{T}\right)^{1/2}$$

where V is the mean molecule velocity, n is the number density, and σ is the collision cross section. The right hand part of the equation gives the pressure and temperature dependence on the billiard ball approximation. Typical half widths at one atmosphere for atmospheric gases are in the range 0.05 to 0.2 cm^{-1} . In general both Doppler and collision broadening are present, and the line shape is in this simple approximation the Voigt profile, which is a convolution of the Doppler and Lorentz profiles. However for the purposes of radiative transfer in the troposphere and lower stratosphere the Lorentz profile is adequate, because $\alpha_L \gg \alpha_D$ at these pressures.

3.2 Transmission of Bands of Spectral Lines

A band of spectral lines is due to one or more vibrational transitions occurring in a particular spectral region, together with rotational structure. The result is usually thousands of spectral lines in a region perhaps 200 cm^{-1} wide. The transmission of such a band can be written down explicitly, but as there are so many lines, it is very time consuming to evaluate. The situation may be further complicated by the possibility of bands of different gases overlapping in the same spectral region. In general the transmission of a spectral interval $\Delta\nu$ is

$$T = \frac{1}{\Delta\nu} \int_{\Delta\nu} d\nu \exp\left\{-\sum_{ij} \int_{z_1}^{z_2} S_{ij}(z) f_{ij} \rho_i(z) dz\right\}$$

where the sum is over all the spectral lines which have significant absorption in the interval $\Delta\nu$. $S_{ij}(z)$ is the strength of the i 'th line of the j 'th absorbing gas evaluated for the physical conditions at position z , $f_{ij}(z)$ is the line shape factor for that line, and $\rho_j(z)$ is the density of the j 'th absorber.

The explicit numerical evaluation of this expression is time consuming not only because there may be thousands of lines involved in the summation, but because lines may be very narrow compared with $\Delta\nu$, and several quadrature points may be required for each line. Approximate methods must be developed for the evaluation of transmission of spectral bands, resulting in various kinds of 'band model'. We will first consider independent spectral lines before going on to the more general case of overlapping lines.

The basic concept in this connexion is the frequency integrated absorption or equivalent width, W , of a spectral line

$$W = \int_{-\infty}^{\infty} A(\nu) d\nu = \int_{-\infty}^{\infty} (1 - T(\nu)) d\nu$$

where $A(\nu)$ is the absorbance. The term 'equivalent width' comes from the astrophysical literature, it is the width of a square sided line with the same area, and zero transmission. The equivalent width of a Lorentz line of strength S and half-width α can be written as

$$W_L = \int_{-\infty}^{\infty} d\nu \left\{ 1 - \exp\left(\frac{Sm}{\pi} \frac{\alpha}{\nu^2 + \alpha^2}\right) \right\}$$

where m is the absorber amount in a homogeneous path. This can be integrated in terms of Bessel functions:

$$W_L = 2\pi\alpha u e^{-u} [I_0(u) + I_1(u)] \text{ where } u = Sm/2\pi\alpha \\ = 2\pi\alpha L(u)$$

Thus defining the Ladenberg-Reiche function $L(u)$. The equivalent width of a Doppler line is

$$W_D = \int_{-\infty}^{\infty} d\nu \left\{ 1 - \exp\left(-\frac{Sm}{\alpha_D/\pi} e^{-\nu^2/\alpha_D^2}\right) \right\}$$

which cannot be integrated in terms of standard functions. However we can define a "Doppler function" $D(w)$:

$$D(w) = \int_0^{\infty} \{1 - \exp(-we^{-x^2})\} dx$$

which can be approximated numerically fairly easily, as it is a function of one variable. Then the Doppler equivalent width is

$$W_D = \alpha_D D(w) \quad w = Sm/\alpha_D\sqrt{\pi}$$

We can define the 'strong' and 'weak' limits of absorption according to whether the optical depth at the line centre is $\gg 1$ or $\ll 1$. In these limits there are useful simple approximations.

The weak limit is independent of line shape, because $\tau(\nu) \ll 1$ for all $\tau(\nu) \ll 1$, and we can write

$$W_{\text{weak}} = \int (1 - e^{-\tau(\nu)}) d\nu = \int \tau(\nu) d\nu$$

$$= \int S m f(\nu) d\nu$$

therefore

$$W_{\text{weak}} = S m$$

as a consequence of the line shape factor being normalised.

The strong limit depends on line shape, and in the case of a Lorentz line we can ignore the term α^2 in the denominator of the line shape, because near the line centre, transmission is zero in any case, and omitting it will not change that, and far enough away from the line centre we can put $\alpha^2 \ll \nu^2$. Thus the strong limit is

$$W_{\text{strong}} = \int (1 - \exp(-\frac{S m x}{\pi \nu^2})) d\nu$$

$$= (\frac{S m x}{\pi})^{1/2} \int (1 - e^{-1/x^2}) dx$$

the integral can be easily evaluated to yield $2\sqrt{\pi}$, so that the equivalent width is

$$W_{\text{strong}} = 2\sqrt{S m x}$$

If spectral lines are independent, then the equivalent width is all that is needed to calculate frequency integrated transmission, because the transmission in a finite interval $\Delta\nu$ is

$$T = 1 - \frac{1}{\Delta\nu} \sum_i W_i$$

where the sum is over all the spectral lines in the interval. Further approximations can be used to evaluate this sum, as we will see later. However, if lines overlap, then we must consider more general band models.

Random Band Models

The concept of a random band model was developed by Goody (1954), who was attempting to model water vapour absorption. The water vapour spectrum, as we have seen, appears to be random both in terms of line positions and line intensities, and we can use this feature to define the general random model, which is the transmission of a spectral interval averaged over all possible rearrangements of line positions.

A very simple derivation of the random model can be carried out with the assumption that the equivalent width of any spectral line is much smaller than $\Delta\nu$, the width of the interval under consideration. In this case we can derive the multiplication property for finite spectral intervals; normally we can only multiply transmissions monochromatically, because they are exponential functions. Consider a spectral interval $\Delta\nu$ having transmission $T_1(\nu)$. If we add an extra line at position ν' , having transmission $T_2(\nu-\nu')$, then the average over the whole interval is

$$\frac{1}{\Delta\nu} \int_{\Delta\nu} T_1(\nu) T_2(\nu-\nu') d\nu$$

If we average this over all possible values of ν' within $\Delta\nu$ the effect of adding one line at random is

$$\bar{T} = \frac{1}{\Delta\nu} \int_{\Delta\nu} \left\{ \frac{1}{\Delta\nu} \int_{\Delta\nu} T_1(\nu) T_2(\nu-\nu') d\nu \right\} d\nu'$$

if we perform the ν' integral first, we obtain

$$\bar{T} = \bar{T}_1 \cdot \bar{T}_2$$

which is the required multiplication property. The mean transmission of a single line of equivalent width w_i can be written

$$T_i = 1 - w_i/\Delta\nu \approx \exp(-w_i/\Delta\nu) \quad \text{if } w_i \ll \Delta\nu$$

Therefore the transmission of N lines placed at random in is $\bar{T} = \prod_i T_i$, or

$$T_i = \exp(-\sum_i w_i/\Delta\nu)$$

which is the general random model.

The same formula can be derived for lines which are not narrow compared with $\Delta\nu$, provided that surrounding spectral intervals are statistically similar. This may easily be seen by considering a small sub-interval within $\Delta\nu$.

We can simplify the random model still further, and avoid having to evaluate a sum over line strengths, by considering strength to be a random quantity as well. If we assume that line strengths are distributed according to a distribution function $N(s)$ then we can write

$$\sum_i w_i = \int_0^\infty N(s) W(s) ds$$

The various random models available differ only in the form used for $N(s)$. There are two conflicting requirements in choosing a distribution function, namely that it should represent reality, and that the integral should be performed algebraically.

The original random model is the Goody Model, which was

$$N(s) = \frac{N_0}{k} e^{-s/k}$$

This distribution has N_0 lines with a mean strength of k , and if we substitute for $N(s)$ and $W(s)$ we obtain

$$\sum_i w_i = \int \frac{N_0}{k} e^{-s/k} \int_0^\infty (1 - e^{-sf(\nu)m}) d\nu ds$$

which can easily be integrated with respect to s to yield

$$\sum_i w_i = N_0 \int \frac{k m f(\nu)}{1 + k m f(\nu)} d\nu = N_0 \int \frac{\tau(\nu)}{1 + \tau(\nu)} d\nu$$

where $\tau(\nu)$ is the optical depth of a line of mean strength. This can clearly be used for a line of arbitrary shape, but is only integrable for some line shapes. For a Lorentz line we obtain

$$\sum_i w_i = \frac{N_0 k m}{(1 + k m / \pi^2)^{1/2}}$$

so that the Goody random model transmission function is

$$T = \exp \left(- \frac{N_0 k m / \Delta\nu}{(1 + k m / \pi^2)^{1/2}} \right)$$

The Godson model uses the distribution

$$\begin{aligned} N(s) &= N_0 / s & s < k \\ &= 0 & s > k \end{aligned}$$

which has N_0 lines between $s = \sigma$ and $s = \sigma/\epsilon$ for $\sigma < k$ which means an infinite number altogether. The result for a



Lorentz line shape can be written in terms of Bessel functions, so is not so simple as the Goody model, but the line distribution is much more realistic. However the Godson model has been somewhat superseded by the Malkmus model, which retains the advantage of a realistic line distribution, whilst producing an algebraically simple transmission function. The line strength distribution is

$$N(s) = \frac{N_0}{S} e^{-s/k}$$

which is actually equivalent to Goody Models combined according to the Godson distribution. The integrals can be performed to give

$$\begin{aligned} \sum W_i &= N_0 \int \log(1 + \tau(\nu)) d\nu \\ &= N_0 \int \frac{\nu}{1 + \tau(\nu)} \frac{d\tau}{d\nu} d\nu \end{aligned}$$

in the case of a general line shape, or in the case of a Lorentz line

$$\sum W_i = 2\pi\alpha N_0 \left\{ \left(1 + km/\pi\alpha \right)^{1/2} - 1 \right\}$$

The strong and weak limits of the Goody and Malkmus models are as follows

	Weak $km/\pi\alpha \ll 1$	Strong $km/\pi\alpha \gg 1$
Goody	$N_0 km$	$N_0 \sqrt{km\pi\alpha}$
Malkmus	$N_0 km$	$2 N_0 \sqrt{km\pi\alpha}$
single line	$S m$	$2 \sqrt{S m \alpha}$

Fig. 12 shows an actual distribution of line strengths for the CO₂ 15 μ band, illustrating the degree to which the models fit real distributions. The distribution models have not been fitted to the actual distribution other than by eye.

Elsassers Regular Model

This is the original band model, and attempts to model the carbon dioxide bands, which have regularly spaced lines.

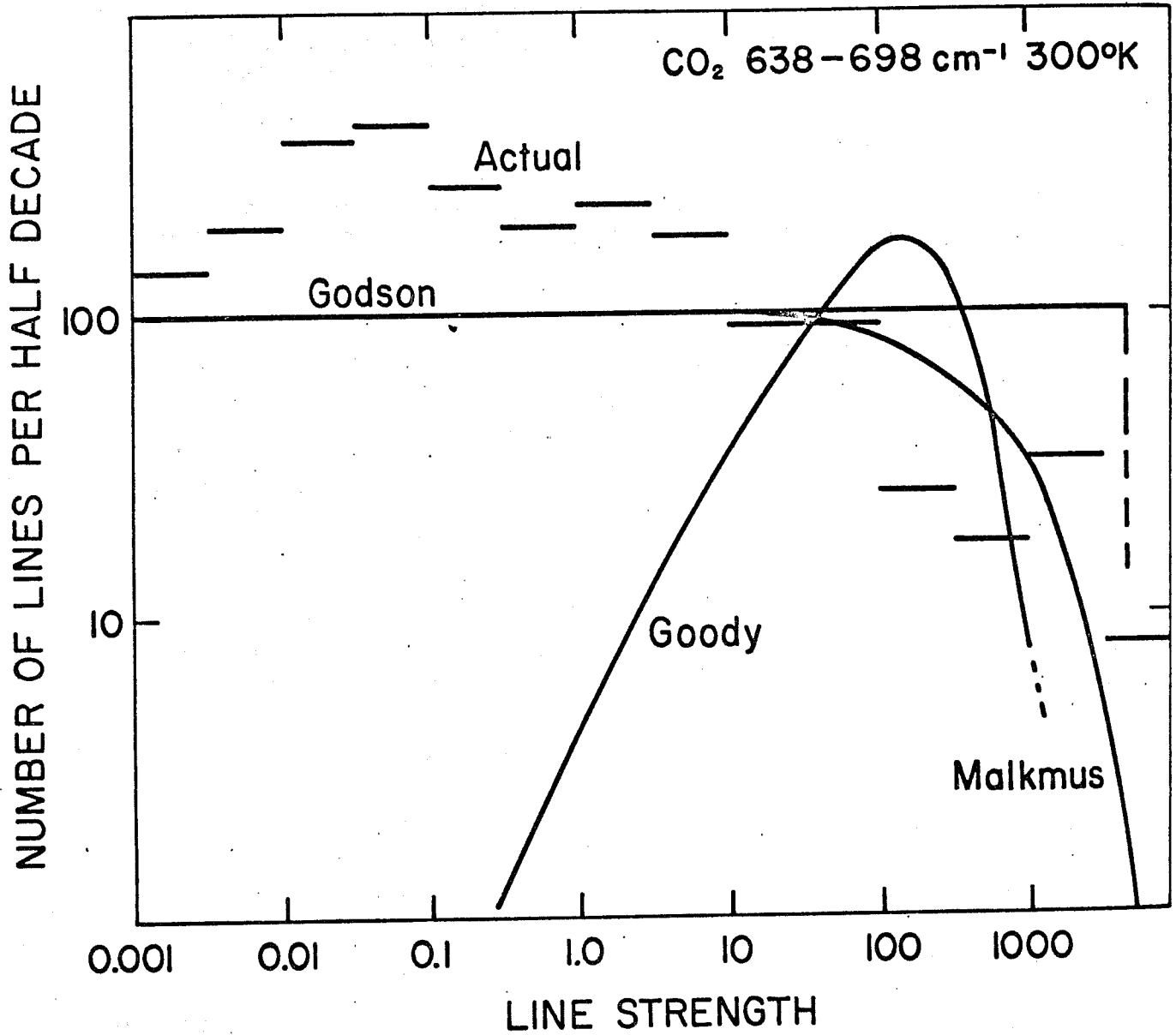
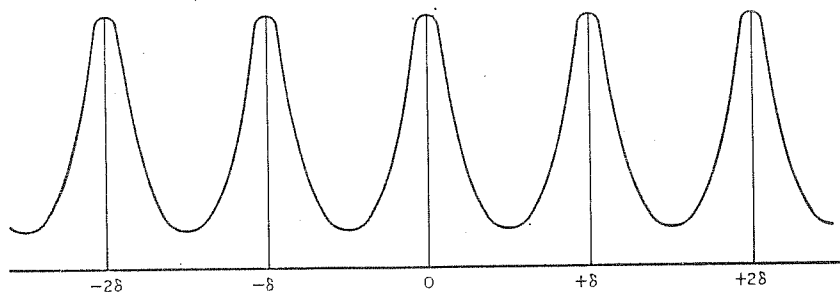


Fig.12 Comparison of an actual distribution of line strengths with the three model distributions. The model distributions have not been fitted to the actual distribution.



An individual element of this array consists of a sum of parts of all lines, and its shape can be shown to be

$$f(\nu) = \frac{\sinh(2\pi\alpha/\delta)}{\cosh(2\pi\alpha/\delta) - \cos(2\pi\nu/\delta)}$$

Transmission of the band is $\exp(-S f(\nu) m)$, which cannot be integrated in wave length in terms of elementary functions but has been tabulated as a function $E(y, u)$ of $y = \alpha/\delta$ and $u = Sm/2\pi\alpha$ in several places, including Goody (1964). The Elsasser function reduces to the Ladenburg-Reiche function in the case of independent lines, and in the limit of strong overlapping lines, i.e. $u \gg 1$, we obtain a version of the square root law:

$$E(y, u) = 1 - \operatorname{erf}(\pi y \sqrt{2u}) = 1 - \operatorname{erf}\left(\frac{1}{\delta} \sqrt{Sm \pi \alpha}\right)$$

where erf is the error function.

We may notice that the strong and weak limit of both the regular and random models can be expressed as functions of the total equivalent width of the lines independently. Yamamoto and Sasamori (1957) have conjectured that in general a good approximation for the transmission of overlapping lines may be

$$T = F\left(\frac{1}{\Delta\nu} \sum_i W_i\right)$$

where the functional form of F depends on the distribution of line positions. In the completely random case it will be exponential, and in the completely regular case it will be a complementary error function. Further credence is given to this conjecture by Goody and Belton (1967) who found that

$$W_{\text{band}} = c \log\left(1 + \frac{1}{c} \sum_i W_i\right)$$

is a good approximation for the total equivalent width of a complete vibration rotation band.

We may find the parameters for band models by two procedures - fitting empirically to laboratory measurements, or fitting to tabulations of line positions and intensities. For example, we may fit the Goody model to laboratory measurements by

manipulating the equation to give

$$1 + \frac{k}{\pi \alpha_0} \cdot \frac{m}{p} = \left(\frac{N_0 k}{\Delta \nu} \cdot \frac{m}{\log T} \right)^2$$

if we plot measurements of $(m/\log T)^2$ against (m/p) we can obtain $k/\pi \alpha_0$ and $N_0 k/\Delta \nu$ from the slope and intercept. As an example of using line intensity data we could fit the Malkmus model by ensuring that $\sum W_i$ is correct in both the strong and weak limits. This gives

Weak: $N_0 k m = \sum_i S_i m$

Strong: $2 N_0 \sqrt{k m \pi \alpha} = 2 \sum_i \sqrt{S_i \alpha_i m}$

so that $N_0 k$ and $N_0^2 k \alpha$ can be derived immediately.

The Angle Integration

In section 2 we defined the flux transmission, and showed that the equation of transfer is the same in the plane parallel case for flux and for intensity. In terms of a band model of some kind, we can write

$$\begin{aligned} T_f(m, p) &= 2 \int_0^{\pi/2} T(m \sec \theta, p) \sin \theta \cos \theta d\theta \\ &= 2 \int_1^{\infty} T(m \sec \theta, p) \frac{d \sec \theta}{\sec^3 \theta} \end{aligned}$$

We can evaluate this integral in certain cases, and use the results to develop an approximation for simplifying the angle integration.

In the case of an exponential (monochromatic) transmission function, $T = e^{-\tau}$, the flux transmission is

$$\begin{aligned} T_f &= 2 \int_1^{\infty} e^{-\tau \mu} d\mu/\mu^3 \quad \text{where } \mu = \sec \theta \\ &= 2Ei_3(\tau) \end{aligned}$$

i.e. the exponential integral of third order.

In the case of the strong limit of a random model we can put $T(mp) = \exp(-(mp)^2)$ so that

$$\begin{aligned} T_f(mp) &= 2 \int_0^{\infty} e^{-(mp)^2 u^2} \frac{du}{u^3} \\ &= 4 \int_0^{\infty} e^{-(mp)^2 y} \frac{dy}{y^3} \\ &= 4 E_5((mp)^2) \end{aligned}$$

i.e. the exponential integral of the fifth order.

A stronger hint is given by the strong limit of a single line, where we will use the equivalent width rather than transmission:

$$W(mp) = \sqrt{emp}$$

$$W_f(mp) = 2 \int_0^{\infty} \sqrt{emp u} \frac{du}{u^3} = \frac{4}{3} \sqrt{emp} = W\left(\frac{16}{9} mp\right)$$

The measured equivalent widths of many vibration-rotation bands have been fitted empirically by a function of the form

$$W(m) = A + B \log(m)$$

If we use this to find an equivalent width for flux we obtain

$$\begin{aligned} W_f(m) &= 2 \int_0^{\infty} A + B \log(mu) \frac{du}{u^3} \\ &= A + 3 \log m + 2B \int_0^{\infty} \log u \frac{du}{u^3} \\ &= A + 3 \log m + B/2 \\ &= A + B \log(e^{3/2} m) \\ &= W(e^{3/2} m) \end{aligned}$$

These results hint that a good approximation for diffuse transmission might be

$$T_f(m) = T(3m)$$

This was first found empirically by Elsasser, who suggested that the diffusion factor $\beta = 1.66$. We see from the empirical band result that $\beta = e^{\frac{1}{2}}$ might be marginally better, as we will integrate over complete bands eventually. Numerical experiment shows that the approximation is good to 1 or 2% for most purposes. For example see Figure 13 for an application to infrared cooling rate.

The Use of Band Models

The aim of the band model is to reduce the work involved in carrying out the frequency integration. The explicit integration of the flux equation:

$$F^\uparrow(z) = \int_0^\infty d\nu F_\nu^\uparrow(\nu, 0) T(\nu, 0, z) + \int_0^\infty d\nu \int_0^{z'} dz' \pi B(\nu, z') \frac{dT(\nu, z, z')}{dz'}$$

is replaced by

$$F^\uparrow(z) = \sum_i \Delta\nu_i F_i^\uparrow(0) T_i(0, z) + \sum_i \Delta\nu_i \int_0^{z'} dz' \pi B(\nu_i, z') \frac{dT_i(z, z')}{dz'}$$

where the finite spectral intervals $\Delta\nu_i$ may be up to 100 or 200 cm^{-1} wide, so that there need only be 10 or 20 of them rather than the thousands of quadrature points needed for $\int d\nu$. The use of band models makes the calculation of radiative transfer feasible, whereas explicit integration is prohibitively expensive.

3.3 Inhomogeneous Paths

The discussion so far has considered only homogeneous paths, in which the absorber is at a constant temperature, pressure and mixing ratio along the path. This is almost never the case in the atmosphere, so we must learn how to calculate or approximate transmission along inhomogeneous paths. This is largely a matter of evaluating the integral for optical depth:

$$\tau(\nu) = \int S(z) f(\nu, z) \rho(z) dz$$

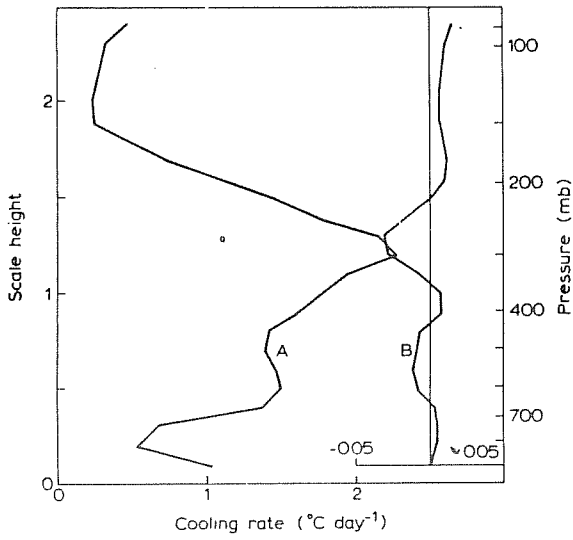


Fig. 13 Accuracy of the diffusivity factor ($1.66 \bar{u}$). A: "exact"; angular integration of transmission functions using the Curtis-Godson approximation. B: "exact" minus "approximate" ("approximate" refers to use of diffusivity factor). (After Rodgers and Walshaw, 1966.)

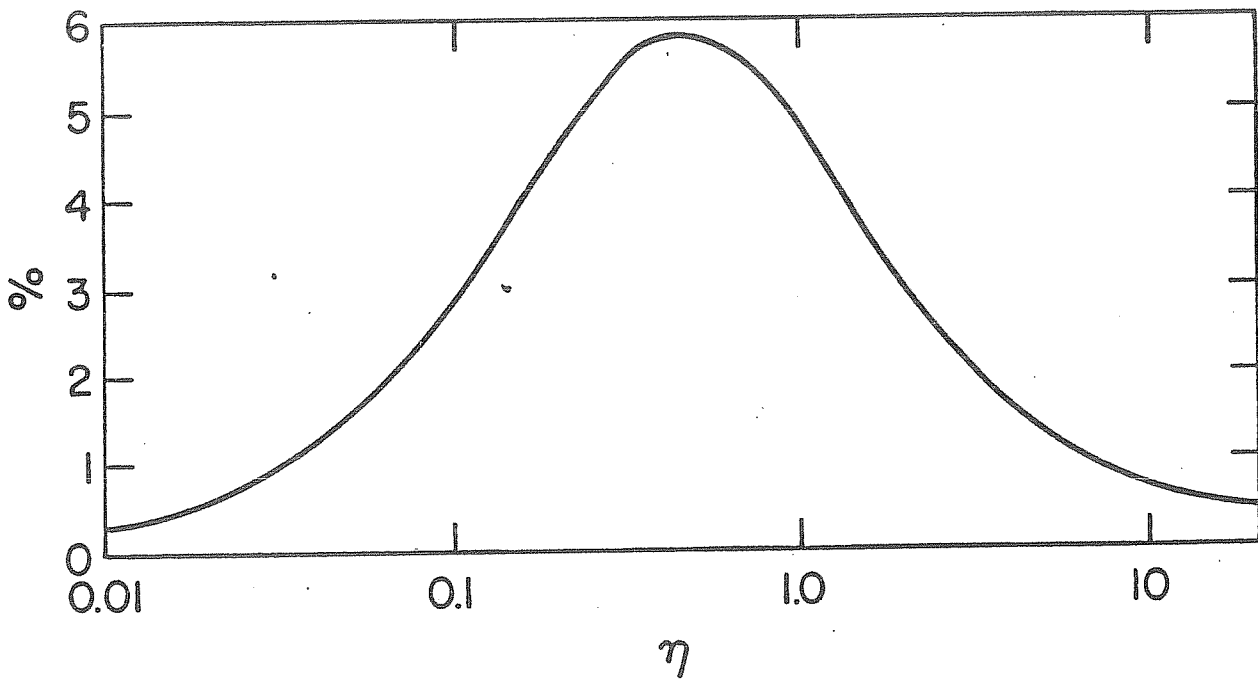


Fig. 14 Percent error in CG approximation for a single line and a constant mixing ratio between $p = 0$ and $p = 1$ atmos.

where the line strength $S(z)$ depends on the temperature at position z , and the line shape factor $f(\nu, z)$ depends on both temperature and pressure.

There is one inhomogeneous case where the integral can be carried out explicitly, and this is for a constant temperature, constant mixing ratio in the vertical (or along a slant path) for the Lorentz line shape. In this case the integral is

$$\tau(\nu) = \int \frac{S}{\pi} \cdot \frac{\alpha_0 p}{\nu^2 + \alpha_0^2 p^2} \rho_0 p dz$$

where p is pressure so that $z = -H \log_e p$. Change the variable to p^2 , and we obtain

$$\begin{aligned} \tau(\nu) &= \frac{S \rho_0 H}{2\pi \alpha_0} \int_{p_1}^{p_2} \frac{\alpha_0^2 dp^2}{\nu^2 + \alpha_0^2 p^2} \\ &= \log_e \left(\frac{\nu^2 + \alpha_0^2 p_1^2}{\nu^2 + \alpha_0^2 p_2^2} \right) \eta \end{aligned} \quad \eta = \frac{S \rho_0 H}{2\pi \alpha_0}$$

The single line transmission function $e^{-\tau}$ is therefore very simple, and the expression can be used in the general line shape version of both the Goody and Malkmus random models, but the conditions are fairly restrictive so that it is of very little use except as a vehicle for testing approximations.

The Curtis-Godson Approximation

The most important approximation for this purpose is the Curtis-Godson approximation, which aims to find a homogeneous path whose transmission approximates the inhomogeneous path under consideration. A homogeneous path may be defined by an absorber mass \bar{m} and a pressure \bar{p} . We try to find expressions for these in terms of the distribution of absorber in the inhomogeneous path, ignoring temperature effects for the moment. We do this by ensuring that the approximation is exact for the equivalent width in both the weak and strong limits of a single line.

The weak limit in the inhomogeneous case is

$$\begin{aligned} W_w &= \int (1 - e^{-\tau}) d\nu = \int \tau d\nu \\ &= \iint S(z) f(\nu, z) dm d\nu \quad \text{where } dm = \rho dz \\ &= \int S(z) dm \end{aligned}$$

The strong limit is a little more complicated, but it can easily be shown to be

$$W_s = 2 \left(\int S(z) \alpha(z) dm \right)^{1/2}$$

The equivalence of the inhomogeneous and homogeneous paths in these limits requires

$$S \bar{m} = \int S dm$$

$$S \alpha \bar{p} \bar{m} = \int S \alpha dm$$

If S and α are not temperature dependent, we obtain the simplest form of the approximation:

$$\bar{m} = \int dm \qquad \bar{m} \bar{p} = \int p dm$$

so that we use the actual absorber mass and a density weighted mean pressure.

A simple test for the accuracy of this approximation is to compare it with the single line constant mixing ratio case described above. Such a comparison is shown in Figure 14 which gives the percentage difference in the equivalent width between the Curtis-Godson approximation and the exact line shape for a path between $p = 0$ and $p = 1$ atm, as a function of η , which happens to be the absorption coefficient at the centre of the homogeneous line, and is the parameter which determines whether a line is strong or weak. The error is small in the weak and strong, as expected, and is largest at $\eta \sim 0.5$. To determine whether this is significant for transfer in the atmosphere we can estimate whether a significant amount of energy is transported by intermediate strength lines. We will see later that the biggest transfer of energy is exchanged with space, and this takes place in a region where transmission is $\sim e^{-1}$. Using the Goody model, this means that $\frac{k \bar{m}}{T} \left(1 + \frac{\eta \bar{m}}{T}\right)^{-1/2} \sim 1$. Combine this with $\eta = \frac{k \bar{m}}{T} \sim 0.5$, and we get $\frac{k \bar{m}}{T} \sim 2$ for the worst case. (δ is the mean line spacing, $\Delta \nu / N_0$). For water vapour and carbon dioxide, $\pi \delta / T \ll 2$ everywhere, but for ozone it is ~ 1 for a path from the surface to space, although it is $\ll 2$ for a path from the stratosphere to space. The Curtis-Godson approximation appears to be adequate for transfer in the atmosphere, except for exchange by ozone from the troposphere to space, and by similar arguments from the stratosphere to the surface.

Temperature dependence of line strength and halfwidth is significant, and should be taken into account. This can be done by a relatively simple extension of the basic Curtis-Godson approximation.

For a single line, the equations give us

$$\bar{m} = \int S(T)/S(T_0) dm \quad \bar{m} \bar{p} = \int \frac{S(T) \alpha(T)}{S(T_0) \alpha(T_0)} p dm$$

where T is temperature and T₀ is any standard temperature. If there are many lines in an interval, the weak limit involves a sum over these lines

$$\sum_i S_i(T_0) \bar{m} = \sum_i \int S_i(T) dm$$

so that

$$\bar{m} = \int \Phi(T) dm \quad \text{where} \quad \Phi(T) = \frac{\sum_i S_i(T)}{\sum_i S_i(T_0)}$$

The effective mass \bar{m} is no longer the actual mass. If it is used for individual lines, they will be in error, but the errors compensate when summed over all the lines. In the strong limit the sum over lines gives

$$\sum_i (S_i(T_0) \alpha_{oi}(T_0) \bar{p} \bar{m})^2 = \sum_i \left(\int S_i(T) \alpha_{oi}(T) p dm \right)^2$$

which cannot be simplified without making some assumptions, because the square root is outside the integral on the right hand side. If there is an approximation of the form

$$\bar{m} \bar{p} = \int \Psi(T) p dm$$

then it must be true for homogeneous paths, which implies that

$$\Psi(T) = \left[\frac{\sum S_i^{1/2}(T) \alpha_i^{1/2}(T)}{\sum S_i^{1/2}(T_0) \alpha_i^{1/2}(T_0)} \right]^2$$

We can also derive this form for inhomogeneous paths by assuming that all the lines have the same temperature dependence. In practise we find by numerical experiment that this temperature dependence is quite satisfactory.

4. Evaluation of Fluxes and Cooling Rates

There are many forms that the transfer equation can be put into, and the proper choice of a suitable form for evaluation depends on the details of the application. There are two basic forms for the flux equation which we will illustrate with the upward component. The equation that we have derived above is the $\int BdT$ form:

$$F^\uparrow(z) = F^\uparrow(0)T(0,z) + \int_0^z \pi B(z') \frac{dT(z',z)}{dz'} dz'$$

This may be integrated by parts to yield

$$F^\uparrow(z) = F^\uparrow(0)T(0,z) + \left[\pi B(z')T(z',z) \right]_0^z - \int_0^z \pi \frac{dB(z')}{dz'} T(z',z) dz'$$

On reordering the terms we obtain the $\int TdB$ form:

$$F^\uparrow(z) = (F^\uparrow(0) - \pi B(0))T(0,z) + \pi B(z) - \int_0^z \pi \frac{dB(z')}{dz'} T(z',z) dz'$$

Of the three terms here, the first one is small because the emission of the surface $F^\uparrow(0)$ is likely to be similar to the Planck function at the temperature of the air at $z = 0$. The second one, $\pi B(z)$, is likely to be the dominant term if the atmosphere is at all optically thick. The equations for downward flux are similar, except that $F^\downarrow(\infty)$ is zero, unless we have an upper boundary such as a cloud which is being treated as a black body.

Any numerical method for the evaluation of flux will reduce to one of the following forms

$$\begin{aligned} F &\approx \sum_i B_i \Delta T_i && + \text{boundary terms} \\ &\approx \sum_i T_i \Delta B_i && \text{"} \\ &\approx \sum_i B_i \frac{dT_i}{dz} \Delta z_i && \text{"} \\ &\approx \sum_i T_i \frac{dB_i}{dz} \Delta z_i && \text{"} \end{aligned}$$

over some set of levels z_i which may be preassigned or chosen dynamically by the ⁱintegration method. It is important to carry out a proper error analysis of a chosen scheme, to ensure that errors are neither too large, so that the wrong answer is reached, or too small, so that computer time is wasted. Care will be needed especially near $z = z^1$, because both $T(z)$ and dT/dz vary very rapidly, especially if absorption is strong.

A simple practical approach to test any numerical integration scheme is to use a transmission function and a Planck function profile for which the integration can be carried out algebraically. For example we could use

$$T(z, z') = \exp(-a|z-z'|) , \quad B(z) = a^2 z$$

so that the numerical method should give

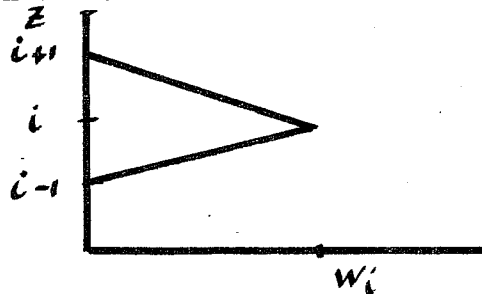
$$F = \int_0^\infty a^2 z e^{-az} dz = 1$$

for any value of a , which is a scaling parameter on z . It can be varied to simulate various strengths of absorption.

A useful trick in the case of an absorber distribution which does not change, such as that of carbon dioxide, is the so called 'Curtis Matrix', which is simply a discretization of the equation of transfer. The concept can be applied to the equation for any radiation quantity, whether intensity, flux or heating rate. If we choose to specify the Planck function at a set of preassigned levels z_i , and use some interpolation rule, then the profile is given by

$$B(z) = \sum_i B(z_i) W_i(z)$$

where $W_i(z)$ is an interpolation function, dependent on the interpolation rule chosen.



For example, linear interpolation makes use of the triangular form indicated in the sketch. The flux equation can then be written in the form (omitting boundary terms):

$$\begin{aligned} F(z_j) &= \int_0^{z_j} B(z') \frac{dT(z_j, z')}{dz'} dz' = \sum_i B(z_i) \int_0^{z_j} \frac{dT(z_j, z')}{dz'} W_i(z') dz' \\ &= \sum_i A_i(z_j) B(z_i) \end{aligned}$$

The matrix $A_i(z_j)$ may be precomputed, and used for a range of different temperature profiles.

A general circulation model or forecasting model normally requires heating rates due to radiation in some appropriate form. There are two basic approaches which may be used, namely flux differences across model layers, or flux gradients at given levels. The choice must depend on the details of the application.

In the case of flux differences, we calculate the change in Net Flux, $F^\uparrow - F^\downarrow$, across a model layer. This will introduce cancellation problems, both in the calculation of net flux, and in differencing it across a layer, especially in regions of strong absorption, where both F^\uparrow and F^\downarrow are nearly $\pi B(z)$, and in regions of weak absorption, where they both change slowly. In the \int BdT form, the integrals from above and below the level have opposite signs, giving a cancellation problem, but in the \int TdB form, a lot of the cancellation is mathematically explicit, because the $\pi B(z)$ term drops out, whilst the integrals have the same sign unless the sign of $\frac{dB}{dz}$ changes. In both cases, cancellation in differencing net flux will be a problem, especially if layers are narrow. If this is the case, it may be that a flux gradient method is more suitable.

In terms of energy units the heating rate is simply

$$h(z) = -\nabla \cdot F = -\frac{dF(z)}{dz}$$

in the case of a plane parallel atmosphere. There are several different formulations, which can be obtained from the two flux equations and by integrating by parts, and by rearrangement of terms.

The most straightforward formulation is obtained by differentiating the \int BdT form

$$h(z) = -F^\uparrow(0) \frac{dT(0,z)}{dz} - 2\pi B(z) \frac{dT(z,z)}{dz} \Big|_{z'=z} - \int_0^z \pi B(z') \frac{dT(z,z')}{dz'} dz'$$

here, the first term is absorption of emission from the surface, the second term is loss due to emission from level z, and the integral is absorption of radiation emitted from all other levels. A little rearrangement gives the "exchange formulation"

$$h(z) = -\pi B(z) \frac{dT(z,0)}{dz} + [\pi B(z) - F^\uparrow(0)] \frac{dT(z,0)}{dz} + \int_0^z [\pi B(z) - \pi B(z')] \frac{dT(z,z')}{dz'} dz'$$

Here, the first term is emission from level z to space, the second term is exchange of radiation with the surface, and the integral is the internal exchange of radiation with all other layers in the atmosphere. Numerical experiment shows that for many situations in the real atmosphere, cooling to space is the dominant form. The internal exchange gives a contribution in the neighbourhood of changes of the lapse rate of temperature, and exchange with the surface only appears to be significant for ozone in the lower stratosphere.

An equivalent to the exchange formulation can be obtained for flux, e.g. for $F^\uparrow(z)$:

$$F^\uparrow(z) = \pi B(z) + \{F^\uparrow(0) - \pi B(z)\} T(0, z) + \int_0^z [\pi B(z') - \pi B(z)] \frac{dT(z, z')}{dz'} dz'$$

A third formulation can be obtained for heating by differentiating the \int TdB version of the flux equation:

$$h(z) = -\pi B(\infty) \frac{dT(z, \infty)}{dz} + [\pi B(0) - F^\uparrow(0)] \frac{dT(z, 0)}{dz} + \int_0^\infty \frac{dT(z, z')}{dz} \frac{dB(z')}{dz'} dz'$$

The terms of this equation cannot be interpreted simply, as can the terms of the other formulations. It has one minor advantage, in that dT/dz is a simpler function to evaluate than $d^2T/dzdz'$, both algebraically and numerically. A fourth formulation can be obtained by integrating by parts to give an integral involving d^2B/dz'^2 , thus relating radiation transfer approximately to diffusion, but the result is of little practical value.

The exchange formulation is probably the most practical, as the dominant term (cooling to space) is a simple one, easily evaluated accurately, and the more complex internal exchange term is relatively small. The integral involves few cancellation problems, as $d^2T/dzdz'$ is negative everywhere.

Emissivity

The use of band models is still too expensive and time consuming for many circulation model applications. There is one more approximation that we can make which will reduce the time needed by an order of magnitude. We can completely eliminate the frequency or wavenumber integration, with the following assumptions

- . only one absorber is involved - i.e. water vapour
- . transmission is a function of one parameter only, for example a pressure scaled water vapour amount, u .

We can then write downward flux for example as

$$F^{\downarrow} = \int_0^{\infty} d\nu \int_{u(z)}^{u(\infty)} du \frac{dA(u, \nu)}{du} \pi B(\theta(u), \nu)$$

where we have used water amount u as the vertical coordinate, rather than height z , and absorption $A=1-T$ in place of transmission T . $\theta(u)$ is the temperature at position u . We can define water vapour emissivity $\epsilon^{\downarrow}(u, \theta)$ as

$$\epsilon^{\downarrow}(u, \theta) = \int d\nu A(u, \nu) \frac{\pi B(\theta, \nu)}{\sigma \theta^4}$$

i.e. a mean absorption weighted by the Planck function. In terms of emissivity, F^{\downarrow} can be written

$$F^{\downarrow} = \int du \frac{d\epsilon(u, \theta(u))}{du} \sigma \theta^4(u)$$

This approximation is really only valid for homogeneous paths; for an atmospheric path it should read:

$$\epsilon^{\downarrow}(z, z') = \int d\nu \int_z^{z'} dz'' \frac{dA(\nu, z, z'')}{dz''} \frac{B(\nu, z'')}{\sigma \theta^4(z'')}$$

but the homogeneous approximation is found by numerical experiment to work remarkably well.

Upward flux must be treated differently, because of the boundary term. If we use the $\int TdB$ formulation, and assume that $\pi B(0) = F^{\uparrow}(0)$ so that one boundary term drops out, then the remaining $\pi B(\nu, z)$ term integrates explicitly over wavenumber, and we are left with the requirement

$$\int d\nu \left(\frac{dB(z')}{dz'} \right) A(\nu, z, z') dz' = \int \sigma \frac{d\theta^4(z)}{dz} \epsilon(z, z') dz'$$

With a homogeneous approximation, as above, this leads to

$$\epsilon^{\uparrow}(u, \theta) = \int d\nu A(u, \nu) \frac{\pi}{\sigma} \frac{dB(\nu, \theta)}{d\theta^4}$$

The accuracy of this kind of approximation is typically 5 Wm^{-2} for flux. Rodgers (1967) has discussed ways in which it might be improved, and finds that the basic technique, using least squares fitting, can give accuracies of more like 1 Wm^{-2} .

Some Simple Approximations

We have commented above that the term in the heating equation which represents cooling to space appears to be dominant. This is illustrated in Fig. 15, taken from Rodgers and Walshaw (1967) for a variety of atmospheric profiles. The main regions where the total cooling differs from cooling to space coincide with changes of the lapse rate of temperature and, in the case of ozone where exchange is taking place between the lower side of the ozone layer and the surface.

The cooling to space expression is

$$h(z) = - \int d\nu \pi B(\nu, z) \frac{d}{dz} T(\nu, z, \infty)$$

which does not require a height integral. This may be simplified in terms of band models as

$$h(z) = - \sum_i \Delta\nu_i \pi B(\nu_i, z) \frac{d}{dz} T_i(z, \infty)$$

or in terms of emissivity as

$$h(z) = \sigma \theta^4(z) \frac{d}{dz} \epsilon(u(z, \infty))$$

which is a very simple approximation, yet of remarkable accuracy. An even simpler approximation is the concept of Newtonian cooling, which is useful in analytic studies of the atmosphere where extreme algebraic simplicity is required. In this case, heating is assumed to be a linear function of the local temperature. We can derive Newtonian cooling coefficients, simply by linearising the cooling to space expression with respect to temperature above some standard temperature, which might be the radiative equilibrium temperature for example:

$$\begin{aligned} h(z) &= \sigma \theta_0^4 \frac{\partial \epsilon}{\partial z} + 4\sigma \theta_0^3 \frac{\partial \epsilon}{\partial z} (\theta - \theta_0) \\ &= h_0(z) + \frac{4h_0(z)}{\theta_0} \cdot (\theta - \theta_0) \end{aligned}$$

We can therefore define a radiative time constant $\tau = \theta_0 / 4h_0(z)$, which is the time scale with which a radiative equilibrium would be approached. In the troposphere typically $h_0 \sim -2 \text{Kday}^{-1}$, so $\tau \sim 260/8 \sim 35$ days, and the stratosphere, $h \sim -8 \text{Kday}^{-1}$, so $\tau \sim 7$ days.

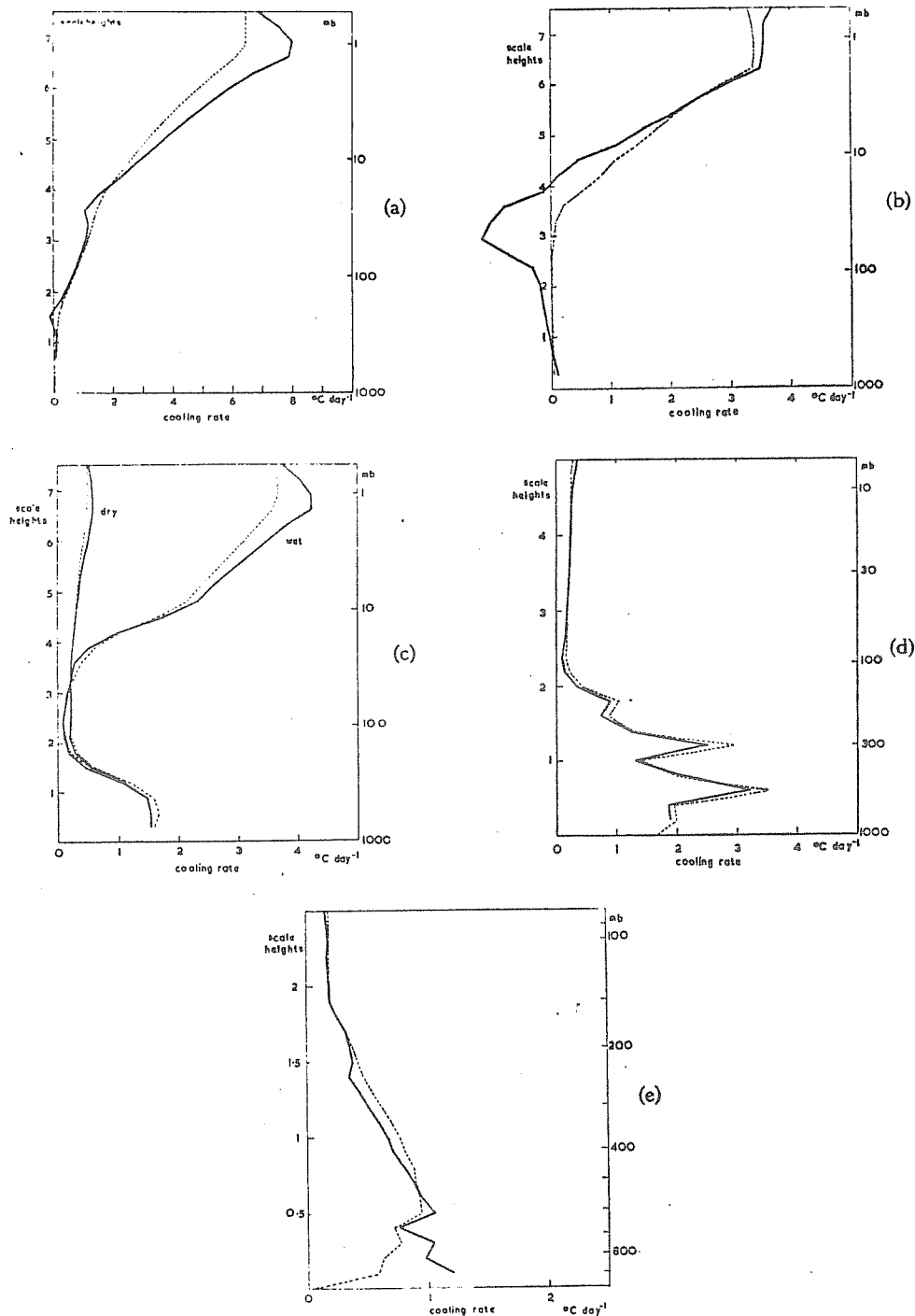


Figure 15. Cooling to space compared with total cooling. — total cooling - - - - - cooling to space.

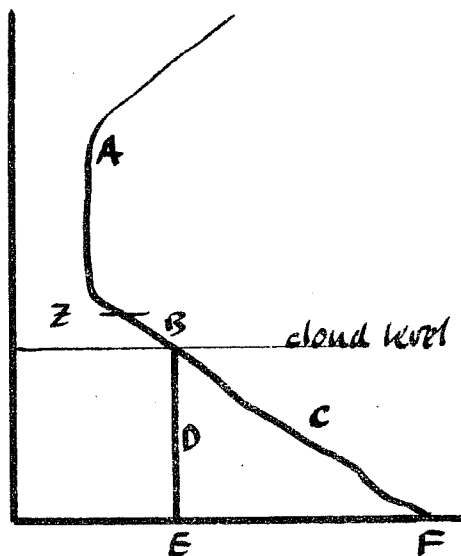
(a) CO_2 15μ band with ARDC temperature profile (Minzner *et al.* 1959). (b) O_3 9.6μ band cooling in a tropical atmosphere (same atmosphere as in Figure 14); (c) wet and dry stratospheres, ARDC temperature profile (Minzner *et al.* 1959); 'dry': mixing ratio at ground 3 g kg^{-1} , falling to 0.002 g kg^{-1} at 150 mb, constant above 150 mb; 'wet': the same up to 90 mb, then increasing to 0.1 g kg^{-1} at 10 mb, constant above 10 mb; (d) cooling due to H_2O in a tropical atmosphere (Nairobi, 7 January 1960); (e) cooling due to H_2O in an arctic atmosphere (Eureka, 11 January 1958, Appendix B).



Cloud

For infrared radiative transfer it is usual to treat cloud as black for two reasons, firstly, it is very nearly black, except in very restrictive circumstances, but more importantly, it is very easy to treat it as black, and very difficult to treat it as a scattering medium. Partially transmitting cloud, e.g. thin cirrus, may be treated in the same way as a partial cloud cover.

The property that allows us to use this approximation is that a black body of temperature θ looks exactly the same radiatively as an infinitely deep region at that temperature. Thus we can treat cloud by modifying the temperature profile. This is illustrated by the diagram. We calculate the



clear sky flux at level z by using the temperature profile and surface temperature given by A-B-C-F, whereas if there is complete cloud cover at the level indicated, we use the profile A-B-D-E, because the cloud looks radiatively like an infinitely deep layer at its surface temperature. Alternatively, we regard the cloud level as the surface.

In the case of partial cloud cover, we take the appropriate linear combination of clear and cloudy cases.

5. Solar Radiation and Scattering

In the absence of scattering, the calculation of solar radiation would be a straightforward matter of calculating transmission by the various absorbers in the ultraviolet, visible and near infrared. Diffuse reflection from the ground and from clouds may be treated using a diffuse transmission function or diffusion factor. However, scattering by dust and haze and Rayleigh scattering in the clear air are significant, as they determine how much solar radiation is reflected back, and hence how much is available for absorption by the atmosphere and surface. The correct treatment of reflection and transmission of solar radiation by clouds is also a matter of scattering.

There are two basic mechanisms, by molecules and by droplets. Scattering by molecules is Rayleigh scattering, and can be regarded as emission by a dipole induced by the incident radiation field. The phase function is that for dipole emission and there is therefore complete polarisation of the scattered radiation at 90° to the incident beam. The frequency dependence of the scattering efficiency is proportional to the fourth power of wave number, and, as is well known, this accounts for the blue colour of the sky. Scattering by droplets is 'Mie Scattering'. and is somewhat more complicated. To treat this, it is necessary to solve Maxwell equations with the appropriate boundary conditions. This has been done for a sphere by Mie, and for a few other simple geometric shapes, but no general solution is available for an arbitrarily shaped scatterer.

We may define an 'efficiency factor' Q for scattering by a particle as the ratio of the cross section σ for scattering to the geometric cross section πa^2 . A sketch of a typical efficiency factor is given in Fig. 16, as a function of the ratio of drop radius to wavelength a/λ . At small values of a/λ the drop behaves as a Rayleigh scatterer, with an efficiency factor proportional to λ^{-4} , whilst at large sizes, the geometric limit, the Q tends to 2, rather than unity as might be expected. One of the two is due to the geometric cross section, the other one is due to diffraction. The oscillations mean that there are values of a/λ for which Q is considerably larger than two. However, in real cases there will be a distribution of drop sizes, and these oscillations will be largely smoothed out. The phase function for spherical drops is similarly oscillating, as may be seen from the typical case shown in Fig. 17, but on averaging over a distribution of sizes, this too is smoothed out, as may be seen from the examples in Fig. 18, which also show some simple analytic functions which have been used as approximate phase functions.

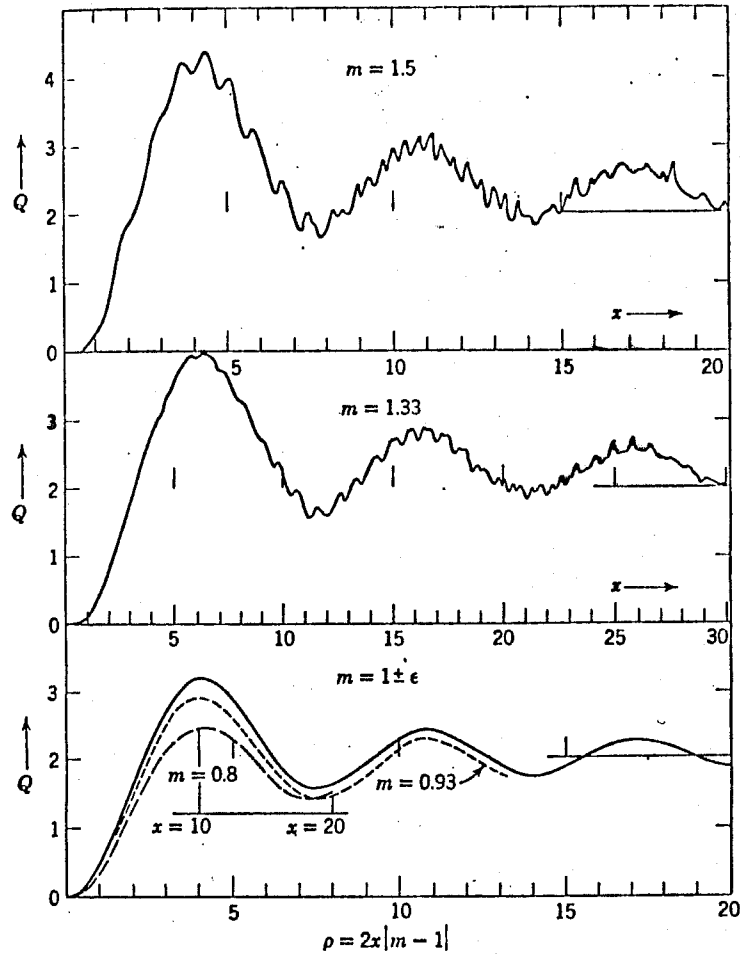


Fig. 16 Extinction curves computed from Mie's formulae for $m = 1.5$, 1.33, 0.93, and 0.8. The scales of x have been chosen in such a manner that the scale of $\rho = 2x|m - 1|$ is common to these four curves and to the extinction curve for $m = 1 \pm \epsilon$. The scale of ρ at the bottom holds for all five graphs. The corresponding scales of x depend on m . For $m = 1.05$ or 0.95 the range shown in the figure corresponds to the range of x from 0 to 200.

(van der Hulst "Light Scattering by Small Particles")

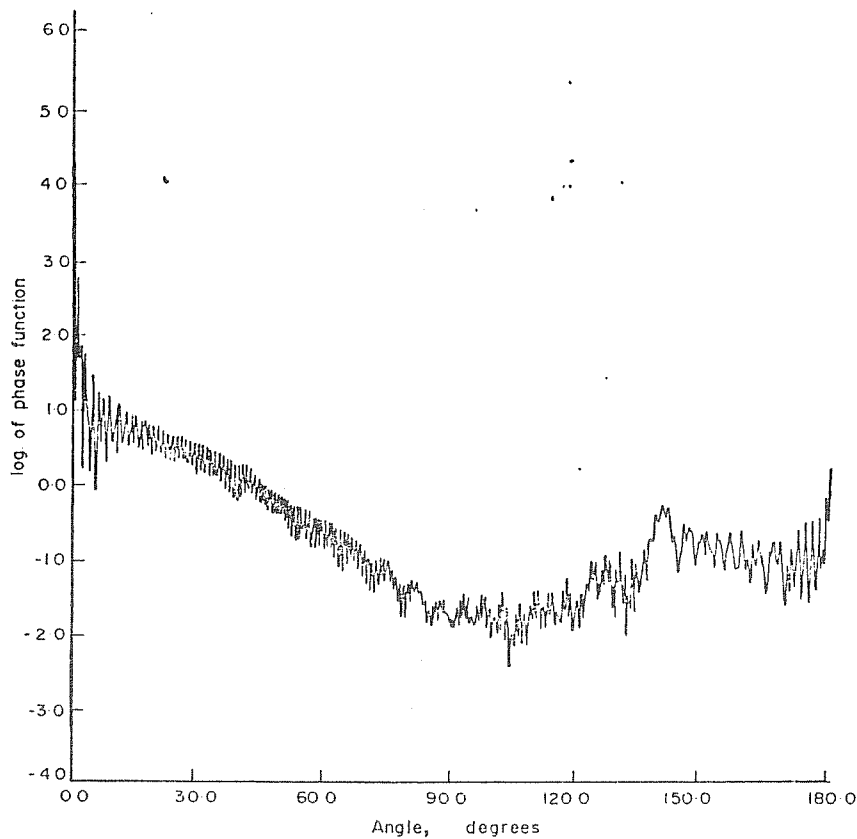


Fig. 17 The phase function for the scattering of radiation at wavelength $\lambda = 0.6 \mu$ by a spherical ice particle of radius 20μ .

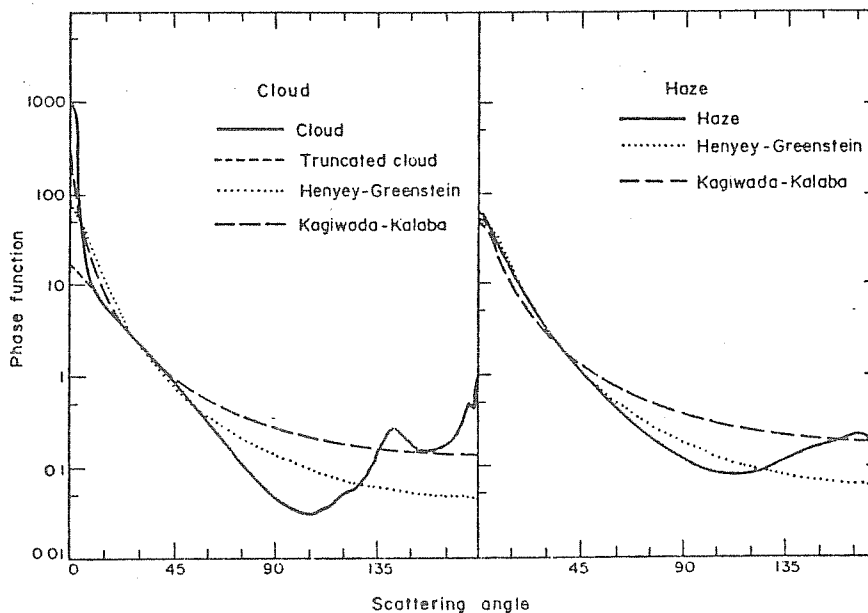


Fig. 18 Phase functions for typical terrestrial cloud and haze particles. Comparison is made with the analytic Henyey-Greenstein, Kagiwada Kalaba and truncated peak analytic phase functions (after HANSON⁽⁵⁰⁾).

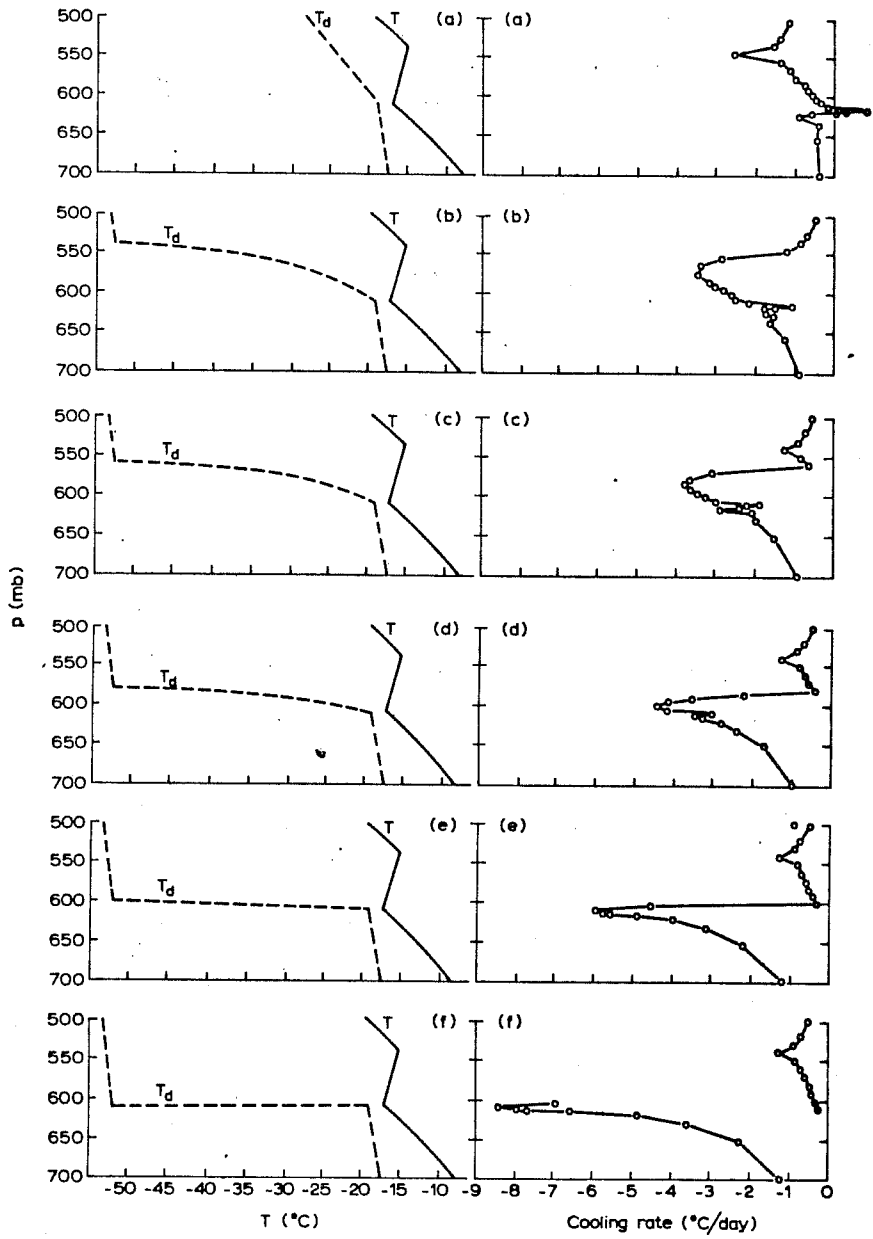
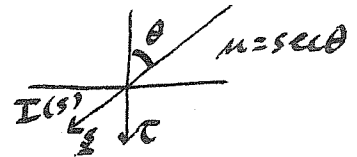


Fig. 19. Radiative cooling profiles associated with a temperature inversion and different degrees of "sharpness" in the water vapour profile. T and T_d are the dry and dewpoint temperatures respectively. (After Staley, 1965.) Paltridge and Platt (1976)

Methods for calculating Scattering

The equation of transfer is

$$\frac{dI(\xi)}{u d\tau} = -I(\xi) + J(\xi)$$

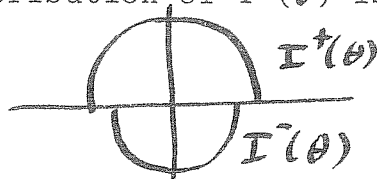


where the source function is made up of an emission component and a scattering component

$$J(\xi) = (1-\omega)B + \omega \int_{4\pi} I(\xi') P(\xi, \xi') d\Omega$$

where $P(\xi, \xi')$ is the phase function for scattering from direction ξ into direction ξ' , and $d\Omega$ is an element of solid angle containing ξ' . The albedo of single scattering is denoted by ω . In the case of isotropic scattering, such as by spherical droplets, $P(\xi, \xi')$ will only depend on the angle between ξ and ξ' .

The simplest approximation for the solution of scattering problems is the two stream approximation, which we have already considered as a simple illustration in section 2. One particular version of the two stream method is the 'Eddington Approximation', in which it is assumed that the upward intensity $I^+(\theta)$ and the downward intensity $I^-(\theta)$ are both independent of θ , so that the angular distribution of $I(\theta)$ is as shown in the sketch.



The resulting phase function will then have four scalar components, P^{++} , P^{+-} , P^{-+} , and P^{--} , rather than being a function of two variables. The problem reduces to the two simultaneous linear equations that we developed in section 2, which can be solved algebraically. The solar beam is usually treated separately, because its angular dependence is not well modelled by this approximation. A further improvement can be obtained by treating the first scattering as a separate term as well, because its angular dependence is likely to be known from the phase function and will be different from the Eddington assumptions. Higher orders of scattering will fit this assumption better. Thus complicated phase functions (e.g. strong forward peaks, rainbows) can be dealt with without too much complication.

Beyond the two stream methods we find ourselves dealing with the more general and complex problem which requires some description of the angular dependence, either in the form of discrete angles or a representation such as Legendre polynomials. Both forms are mathematically identical, it is simply the representation which is different. For purposes of flux calculations, we are not interested in the azimuth dependence of intensity, so we can average the transfer equation azimuthally, thus reducing the dimensionality of the problem. Let us model the radiation field in terms of the azimuth average of intensity in a finite set of directions θ_i , $i = 1 \dots N$, (or, equivalently in terms of a finite number of Legendre polynomial coefficients). Then the

phase function becomes a matrix $P_{ij} = P(\theta_i, \theta_j)$, and the intensity is a vector \underline{I} with components I_j . The source function is

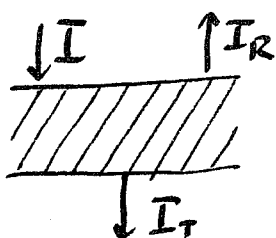
$$J_i = (1-\omega)B + \omega \sum_j P_{ij} I_j$$

and the equation of transfer becomes

$$\frac{1}{\mu_i} \frac{dI_i}{d\tau} = -I_i + (1-\omega)B + \omega \sum_j P_{ij} I_j$$

which is a set of N first order linear differential equations, which can in principle be solved explicitly by well known methods. There are some technical problems in connexion with discretising the phase function, especially if it has a large forward peak, as many do, but we will not treat that question here. A recent paper by Wiscombe (1977) is worth consulting in this respect.

In many contexts the concept of transmission and reflection operators is a very useful one, being a generalisation of the scalar transmission and reflection functions to a set of N angles or polynomial coefficients. For this purpose we will ignore thermal emission.

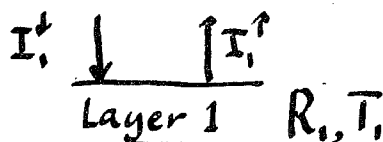


Consider a field of radiation represented by the intensity vector \underline{I} incident on a plane parallel scattering layer. Let the reflected intensity be \underline{I}_R and the transmitted intensity be \underline{I}_T . Then, because the equation of transfer is linear, we can define a reflection matrix or operator \underline{R} and a transmission matrix or operator \underline{T} , such that

R_{ij} is the reflection coefficient from angle j to angle i :

$$\underline{I}_T = \underline{T} \cdot \underline{I} \quad \underline{I}_R = \underline{R} \cdot \underline{I}$$

The matrices \underline{T} and \underline{R} can be easily computed for very thin layers, where we can make the assumption of single scattering. For thick layers, we can make use of the rules of combination to build up from thin layers. For example, consider the two arbitrary (thick or thin) layers in the sketch, with transmission and reflection matrices T_1, R_1 and T_2, R_2 respectively. A little algebra shows how we can define transmission and reflection operators for the combined layer.



The intensity I_2^\downarrow due to transmission of I_1^\downarrow and reflection of I_2^\uparrow :

$$I_2^\downarrow = T_1 I_1^\downarrow + R_1 I_2^\uparrow$$

The intensity I_2^{\uparrow} is due to reflection of I_2^{\downarrow}

$$I_2^{\uparrow} = R_2 I_2^{\downarrow}$$

Therefore

$$I_1^{\downarrow} = T_1 I_1^{\downarrow} + R_1 R_2 I_2^{\downarrow}$$

On rearrangement

$$I_1^{\downarrow} = (1 - R_1 R_2)^{-1} T_1 I_1^{\downarrow}$$

where the inverse is a matrix inverse.

Then the intensity transmitted by the combined layer is

$$I_3^{\downarrow} = T_2 I_2^{\downarrow} = T_2 (1 - R_1 R_2)^{-1} T_1 I_1^{\downarrow}$$

And the combined transmission matrix is therefore

$$T_{12} = T_2 (1 - R_1 R_2)^{-1} T_1$$

Similarly we may derive the combined reflection matrix

$$R_{12} = R_1 + T_1 R_2 (1 - R_1 R_2)^{-1} T_1$$

In the case of no absorption, $T + R = 1$, and the equations can be simplified a little. These rules of combination and the basis of the 'doubling method', where an elementary layer is built up to a finite thickness very efficiently by successively doubling its thickness.

Scattering and Line Absorption

One of the major problems of applying scattering calculations to the real atmosphere is the problem of line absorption. All the scattering methods have been developed for absorbers with a constant absorption coefficient, i.e. grey absorbers. We could in principle carry out an explicit integration over frequency, but this would be several orders of magnitude more expensive even than explicit integration in the case of no scattering.

Basically the problem is that scattering methods require that scalar transmissions can be multiplied, i.e. they are exponential, and this is not true for band models. Band models, on the other hand, require that all photons travelling from A to B do so along the same path. This is not true for scattering.

Two approximations have been developed to cope with this problem, the so-called 'picket fence' model, and the 'photon path distribution' method. In the picket fence model, the transmission is approximated by a sum of exponential absorptions :

$$T = \sum_1^N a_i \exp(-k_i m)$$

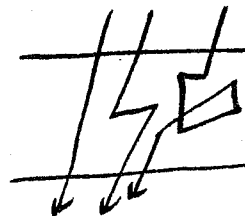
and the scattering calculation is carried out N times, once for each term in the sum. This can only be used if the transmission function can be expressed as a function of a single scaled absorber amount, so that the sum of exponential forms can be fitted.

In the photon path distribution method, scattering calculations are used to determine the distribution $N(\ell)$ of path lengths traversed by photons, so that band models may be used for the transmission of a layer:

$$T_{\text{layer}} = \int N(\ell) T_{\text{gas}}(\ell m, \rho) d\ell$$

or in a discrete form:

$$T_{\text{layer}} \approx \sum_{\ell} N(\ell) T_{\text{gas}}(\ell m, \rho) d\ell$$



There is a kind of duality between these two methods. If we use the continuous form of the picket fence model

$$T_{\text{gas}}(m) = \int a(k) e^{-km} dk$$

and we see that the distribution of absorption coefficients is an inverse Laplace transform of the transmission function. If we substitute this into the photon path distribution we find that the transmission of a layer is

$$T = \int d\ell \int dk a(k) N(\ell) e^{-k\ell m}$$

which is symmetrical in absorption coefficient k and path length ℓ . Thus to find the distribution of absorption coefficients $a(k)$ we do a numerical inverse Laplace transform of $T_{\text{gas}}(m)$ and to find the distribution of photon paths, we do a numerical inverse Laplace transform of the transmission of a layer for an exponential absorption.

In practise we find that we can do better with fewer terms, even with one term, with the photon path method, than we can do with the picket fence model. This property is clearly related in some way to the use of $\beta = 1.66$ for diffuse radiation, which is of course a particular case of a photon path distribution.

6. Parameterization

The whole purpose of this course is to discuss methods of calculating radiative transfer in various kinds of numerical forecasting models, and in this context there are a number of restrictions and difficulties which must be taken into account. The quantities that are normally required are the gains and losses of energy by radiation in model boxes and for surface areas. The time available for the computation is small, it should be smaller than the time taken by the dynamical part of the calculation, so that accurate approximation is paramount.

The main problem in accurate computation is probably that the atmosphere is not specified as well as we might wish. The input data is typically the three velocities, temperature and humidity at the surface and for a relatively small number of levels, on a coarse horizontal grid. As a consequence of the limitations of numerical models, we are left with several problems:

- 1) All the quantities needed for radiative transfer calculations are not specified, and in particular :cloud, aerosol, ozone, and surface spectral properties.
- 2) The vertical resolution is too coarse. Profiles of temperature and humidity are not well enough defined, so that we must find some suitable interpolation.
- 3) The requirement of speed implies that a high level of approximation must be used, and this determines the types of interactions which will be ignored.

Cloud parameterization

This is probably the largest unsolved problem in the subject of radiative transfer in numerical models. The problem is not so much one of calculating radiation transfer in the presence of a given cloud, but rather one of determining where clouds are, given only fields of velocity, temperature and humidity. There is even a certain amount of difficulty defining what a cloud is, as you may imagine by looking at the sky when there are broken clouds with ill-defined edges. Furthermore, clouds are definitely a subgrid scale phenomenon, and the variation over a model box can be enormous.

The simplest radiative model of a cloud is as a black body if there is any absorption, for example in the infrared, and as a diffuse reflector if there is none, for example in the visible. In general, radiative calculations treating scattering property are too slow for use in forecasting and general circulation models. To define a cloud for radiative purposes we need : cloud top height, cloud base height and spectral properties in the case of stratus cloud, and for other types we also need a three-dimensional distribution, in principle, although it is difficult to use in practise. It is probably best to regard a 3-D cloud distribution as a broken layered cloud at the present state of development of the theory. However, we are not given any of these quantities by the typical forecasting model, so that it is necessary to parameterize cloud as well as radiation. To do this correctly, we need a physical understanding of the formation of cloud of all types. The mechanisms are different for the different types of cloud, e.g. cirrus, altostratus, frontal stratus, cumulus, cumulonimbus all form in completely different contexts. Furthermore, a cloud

can be radiatively active long after it has ceased to be significant for rainfall or latent heat release, so that we must treat not only the formation of cloud, but its advection and dissipation. Very little work has been done on this aspect of cloud modelling, and probably the only cloud parameterization that has been at all widely used is an empirical relationship due to Smagorinsky between mean cloud amount in model boxes and relative humidity. Clearly cloud should also depend on such things as vertical and horizontal velocities, and on a radiation.

Aerosol

Not enough is known about aerosol for a successful parameterization, and probably not enough is known even to establish a climatology with any useful degree of accuracy. However, aerosol is a minor effect, and can be allowed for to first order (as can all unknown effects) by "tuning" the radiation parameterization to fit such global criteria as overall radiation balance, and agreement with satellite measurements.

Ozone

The mechanisms of production, transport and destruction are moderately well understood, and can be modelled. However, it is too complex a calculation for forecasting models, and only affects the stratosphere significantly. A climatological ozone distribution is quite adequate for forecasting purposes.

Surface

Surface albedo is required to calculate reflected solar radiation, and surface emissivity is needed to calculate emission of thermal radiation. These quantities depend to a certain extent on geography, and can be pretabulated, but there are also variations which depend on the weather. The most important is probably snow and ice cover, which dramatically affects albedo, but in a way which depends on vegetation. Minor changes are also due to rainfall, the annual cycle of growth of vegetation, and agriculture. They could all be parameterized, but probably only the snow/ice change is significant.

Resolution

As we have already said, the question of poor model resolution gives us the interpolation problem of determining which continuous temperature profile should be used to calculate radiation. There is also the related question of subgrid scale processes which will be lost. For example radiation will tend to erode

discontinuities of temperature gradient such as the tropopause, surface inversions, and fronts. There are cases in which the converse is true, depending on the humidity profile. For example Staley (1965) has studied radiative cooling in 'dry inversions', and found that they are likely to be maintained by radiation (See Fig. 19). This kind of interaction will be lost if low vertical resolution is used. Conversely, if the wrong kind of interpolation is used, e.g. linear interpolation, then calculated flux divergence will give very misleading cooling rates at the given levels, because there will be lapse rate discontinuities. In this case, flux differences rather than divergences are more appropriate.

A question that is worth studying with regard to resolution problems is as follows: Given the model's values of T_i at a small number of levels z_i , what real atmospheric profile $T(z)$ does this represent? A good answer to this question would increase accuracy of calculated radiative transfer at very little cost.

Computational Speed

It is this requirement which finally decides which kind of parameterization will be used. The more time available, the better the calculation can be, so that there is a trade off between speed and accuracy.

Any method should be capable of getting the climatology right, getting the interactions right, and being tuned if necessary.

Getting the climatology right means that there will not be overall systematic errors, so that the model atmosphere will be driven by the right energy input in the right places. It should be possible to compute the known radiation climatology given the known temperature, humidity and cloud climatology, particularly with regard to global radiative balance and the equator to pole gradient of radiation. It should also be possible to reproduce the radiation climatology from a "climate run" at the general circulation model, but this is much more difficult, and involves a test of the whole model.

Getting the interactions right means essentially that the change in radiation due to a change in an input parameter should be correct, and that all the relevant inputs should be interactive. The interactions are, for example, the dependence of cooling temperature and humidity, the greenhouse effect, the water vapour greenhouse amplification, snow/ice line feedback, etc.

Capability of being tuned if necessary is simply a matter of adjusting parameters within reasonable bounds so that the climatology requirements and the interaction requirements are satisfied.

It is possible to write down a rough hierarchy of calculation methods, in order of increasing complexity, both for solar and terrestrial radiation:

S O L A R	T E R R E S T R I A L	
Surface temperature specified	Constant cooling rate	non- inter- active
Climatology	Climatology	
Absorption function-fixed function of absorber amount	Newtonian cooling	capable of being used interactively
	Cooling to space	
2 Stream Scattering methods	Emissivity	
	Band Models	

APPENDIX - Stages of Approximation

We list here in summary the stages of approximation we can go through from the explicit formulation down to Newtonian Cooling. This is illustrated in terms of downward flux for all except the last two stages, but the same simplifications also apply to other quantities:

EXPLICIT INTEGRATION

$$F^{\downarrow}(z) = \int_0^{\infty} d\nu \int_0^{\infty} \frac{d\mu}{\mu^3} \int_z^{\infty} dz' \pi B(\nu, z') \frac{d}{dz'} \exp\left\{-\mu \int_z^{\infty} \sum_i S_i(z'') f_i(\nu, z'') \rho(z'') dz''\right\}$$

CURTIS GODSON APPROXIMATION

Replace the integral over z'' by a homogeneous path

$$F^{\downarrow}(z) = \int_0^{\infty} d\nu \int_0^{\infty} \frac{d\mu}{\mu^3} \int_z^{\infty} dz' \pi B(\nu, z') \frac{d}{dz'} \exp\left\{-\mu \sum_i \bar{S}_i f_i(\nu, \bar{\rho}) \bar{m}\right\}$$

BAND MODEL

Replace the integral over wave number of the exponential by a band model and a sum over wave number

$$F^{\downarrow}(z) = \sum_j \Delta\nu_j \int_0^{\infty} \frac{d\mu}{\mu^3} \int_z^{\infty} dz' \pi B(\nu_j, z') \frac{d}{dz'} T(\nu_j, \mu \bar{m}, \bar{\rho})$$

DIFFUSE APPROXIMATION

Replace the integral over angle ($\mu = \mu \theta$) by a diffusion factor β

$$F^{\downarrow}(z) = \sum_j \Delta\nu_j \int_z^{\infty} dz' \pi B(\nu_j, z') \frac{d}{dz'} T(\nu_j, \beta \bar{m}, \bar{\rho})$$

EMISSIVITY

Replace the sum over wave number and the transmission by an emissivity

$$F^{\downarrow}(z) = - \int_z^{\infty} dz' \sigma \theta^{\downarrow}(z') \frac{d\mathcal{E}(u(z, z'))}{dz'} \quad u = \beta \bar{m} \bar{\rho} \quad \text{perhaps}$$

COOLING TO SPACE

Ignore all terms in the heating equation, apart from the exchange with space term

$$h(z) = \sigma \theta^{\downarrow}(z) \frac{d\mathcal{E}(z, \infty)}{dz'}$$

NEWTONIAN COOLING

Linearise cooling to space with respect to temperatures and take $d\mathcal{E}/dz'$ as constant

$$h(z) = a + b\theta(z)$$

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