INTERNAL DISSIPATION IN THE ATMOSPHERE

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1. Observational Studies

This and the two lectures following are concerned with the turbulent dissipation of kinetic energy in the atmosphere, away from the boundary layer and with the representation of this process in numerical models for forecasting.
In this lecture we concentrate mainly on the observed dissipation in the atmosphere.
To begin the discussion and to provide a framework for the observational studies we consider the kinetic energy budget for the GFDL model as discussed by Smagorinsky et al (1965) (hereafter denoted by S.) The physical parameterizations included in the model included a detailed radiation calculation, surface energy and momentum exchanges, a boundary layer and internal dissipation. Moisture was not treated explicitly but the stabilising effect of moist convection was included implicitly by requiring an adjustment of the lapse rate whenever it exceeded the moist adiabatic layer.

The equations of motion and continuity in pressure coordinates are

$$\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla\underline{u} + \omega\frac{\partial \underline{u}}{\partial p} - f\underline{k}\times\underline{V} = -\nabla\phi + \underline{F} \qquad 1$$

$$\nabla.\underline{u} + \frac{\partial\omega}{\partial p} = 0 \qquad 2$$

With $\underline{K} = \frac{1}{2}\underline{U}^2$ we have, using the continuity equation

$$\frac{\partial K}{\partial t} + \nabla.\underline{u}K + \frac{\partial}{\partial p}\omega K = -\underline{u}.\nabla\phi + \underline{u}.\underline{F} \qquad 3$$

Now $\underline{U}.\nabla\phi = \left(\nabla.\phi\underline{u} + \frac{\partial}{\partial p}\omega\phi\right) - \phi\left(\nabla.\underline{u} + \frac{\partial\omega}{\partial p}\right) - \omega\frac{\partial\phi}{\partial p}$

$$= \nabla.\phi\underline{u} + \frac{\partial}{\partial p}\omega\phi + \omega\alpha; \qquad \alpha = RT/p = -\frac{\partial\phi}{\partial p}$$

Hence

$$\frac{\partial K}{\partial t} = -\nabla.\underline{u}K - \frac{\partial}{\partial p}\omega K - \nabla.\phi\underline{u} - \frac{\partial}{\partial p}\omega\phi - \omega\alpha + \underline{u}.\underline{F} \qquad 4$$

If we take an area mean and assume no flow across the horizontal boundaries we get

$$\frac{\partial}{\partial t}\overline{K}^H = -\frac{\partial}{\partial p}\overline{\omega K}^H - \frac{\partial}{\partial p}\overline{\omega\phi}^H - \overline{\omega\alpha}^H + \overline{\underline{F}.\underline{u}}^H \qquad 5$$

and we have the identity

$$\overline{\underline{u}.\nabla\phi}^H = \frac{\partial}{\partial p}\overline{\omega\phi}^H + \overline{\omega\alpha}^H$$

We shall speak of $\underline{U}.\nabla\phi$ as the source term for kinetic energy and $F.\underline{U}$ as the sink. The terms $\nabla.\phi\underline{u}$, $\frac{\partial}{\partial p}\phi\omega$ we shall speak of as pressure work terms. The divergence of the correlations between velocity and the "pressure" in this system are a means of transmitting kinetic energy that is quite different from the advection terms.

$-\overline{\omega\alpha}^H$ will be called the energy conversion term as it represents the conversion of potential plus internal energy into kinetic energy. It is positive if in the area mean, warm air is rising and cold air sinking ($\omega < 0$ when α is low, $\omega > 0$ when α is high).

Fig. 1 (from S) shows a time and hemispherically averaged picture of the terms in Equation 5, in the model after it had achieved a quasi steady state.
The energy conversion is large in mid levels and falls to zero near the surface and to small values near the tropopause.
The dissipation is largest where the energy conversion is smallest i.e. near the boundary layer and near the tropopause. The overall balance between these processes is maintained not by the advection of kinetic energy away from the region of large conversion, but rather by the pressure work term.
Fig. 2 (from S) shows the source term $-\overline{\mathbf{u}\cdot\nabla\phi}^H$, the conversion term and the term $-\overline{\omega\phi}^H$. The source and sink terms almost balance exactly in the boundary layer and at the jet stream level just below the tropopause.

On the other hand, the pressure work term and the conversion term are of nearly equal magnitude and opposite sign in mid troposphere. Thus although the mid troposphere is the main area of conversion of potential to kinetic energy, the source term for kinetic energy $\overline{\mathbf{u}\cdot\nabla\phi}^H$ is almost zero there. In this experiment the ratio of the dissipation above 811 mb to that below was 1.6 to 1.

Before leaving the model results we look briefly at the eddy and zonal kinetic energy budgets in the model. Let $\overline{(\quad)}^\lambda$ denote a zonal average and $(\quad)'$ a departure from it.
Define

$$K_E = \tfrac{1}{2}\left[(u')^2 + (v')^2\right]$$

$$K_Z = \tfrac{1}{2}\left[(\bar{u}^\lambda)^2 + (\bar{v}^\lambda)^2\right]$$

then

$$\frac{\partial}{\partial t}\overline{K_E}^H = -\frac{\partial}{\partial p}\overline{\omega K_E}^H - \overline{\omega'\alpha'}^H - \frac{\partial}{\partial p}\overline{\omega'\phi'}^H + \overline{\mathbf{u}'\cdot\mathbf{F}'}^H + (K_Z\cdot K_E)_1 + (K_Z\cdot K_E)_2 \qquad 6$$

$$\frac{\partial}{\partial t}\overline{K_Z}^H = -\frac{\partial}{\partial p}\overline{\omega K_Z}^H - \overline{\bar{\omega}^\lambda\bar{\alpha}^\lambda}^H - \frac{\partial}{\partial p}\overline{\bar{\omega}^\lambda\bar{\phi}^\lambda}^H + \overline{\bar{\mathbf{u}}^\lambda\cdot\bar{\mathbf{F}}^\lambda}^H - (K_Z\cdot K_E)_1 - (K_Z\cdot K_E)_2 \qquad 7$$

$$(K_Z\cdot K_E)_1 \approx -\overline{\overline{u'v'}^\lambda\,\frac{\cos\theta}{a}\frac{\partial}{\partial\theta}\frac{\bar{u}^\lambda}{\cos\theta}}^H - \overline{\overline{\omega' u'}^\lambda\,\frac{\partial}{\partial p}\bar{u}^\lambda}^H$$

$$(K_Z\cdot K_E)_2 = \frac{\partial}{\partial p}\overline{\bar{u}^\lambda\,\overline{u'\omega'}^\lambda}^H$$

Fig. 3 (also from S) shows the distribution in the vertical of these terms in the model. The term $\overline{\omega'\alpha'}^H$ is much the larger of the two conversion terms. The eddy kinetic energy is exported to the boundary layer and to the jet stream level, in both of which areas it is dissipated. In addition there is the important conversion of eddy to zonal kinetic energy at the jet stream level, which drives the zonal flow. The zonal kinetic energy is exported to the lower troposphere where it too is subject to dissipation.

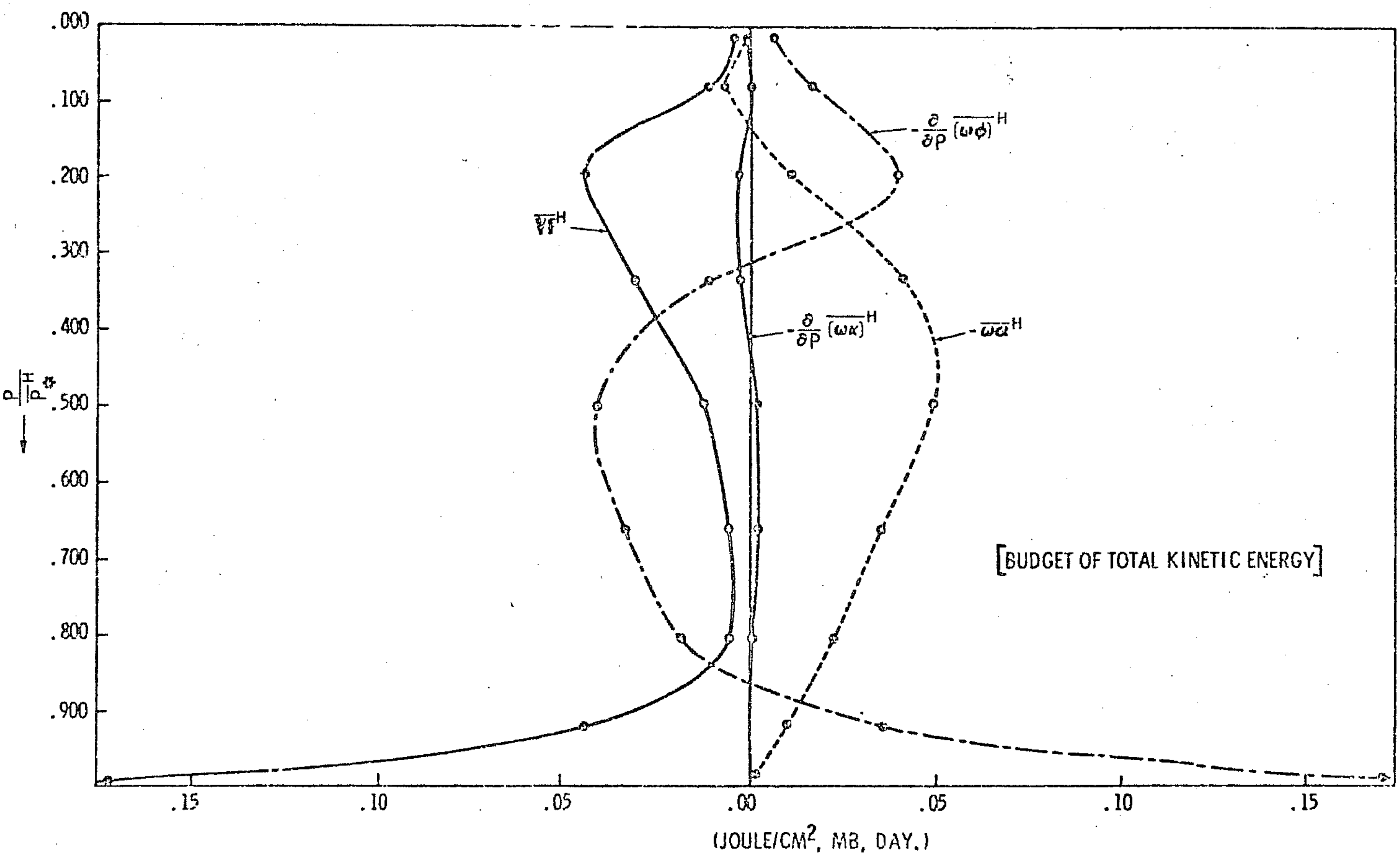


Fig. 1—Vertical distributions of the rate of change of kinetic energy on each isobaric surface due to various mechanisms.

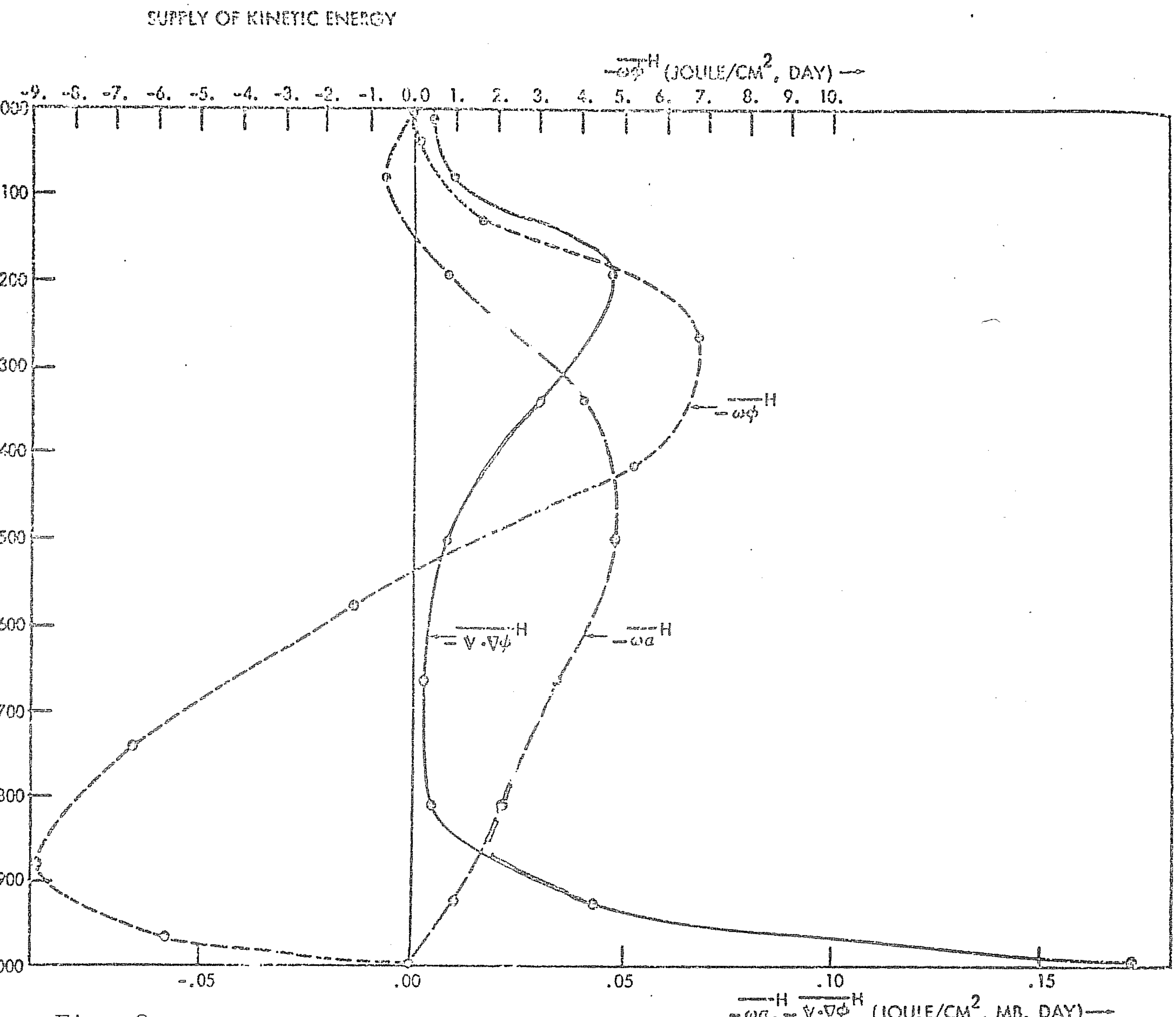


Fig. 2

--Vertical distribution of $-\overline{\mathbf{V}\cdot\nabla\phi}^H$, $-\overline{\omega\phi}^H$, and $-\overline{\omega\alpha}^H$ on an isobaric surface.

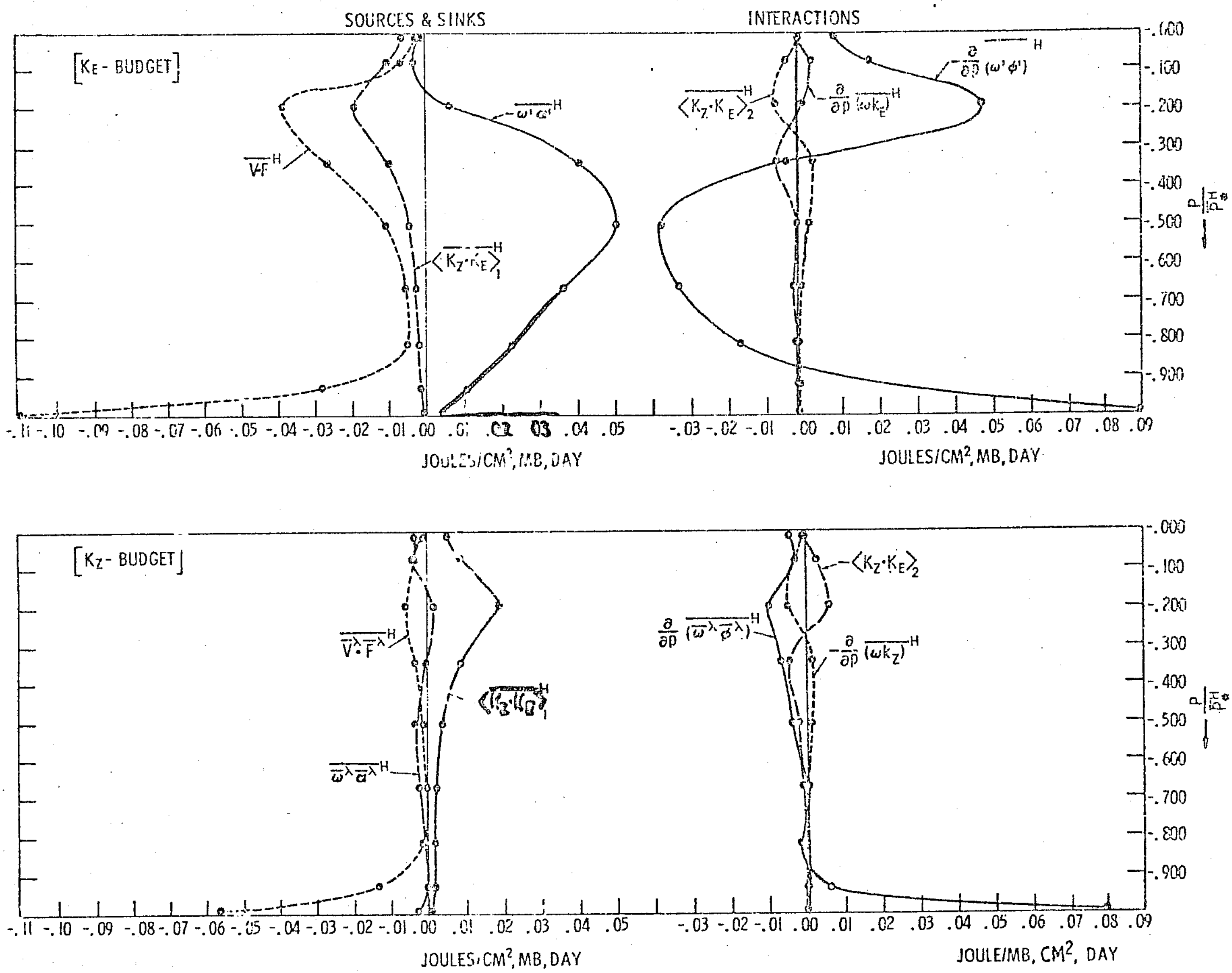


FIGURE 3 —The vertical distributions of the rate of change of eddy kinetic energy due to various terms on an isobaric surface are shown in the upper part of the figure, and those of the rate of change of zonal kinetic energy are shown in the lower part.

Fig. 4 shows the vertical profiles of the source terms $-\overline{\underline{u}^{a}\cdot\nabla\phi^{a}}$ and $\overline{\underline{u}'\cdot\nabla\phi'}$ which are of course the sums of the respective conversion and pressure work terms. A surprising feature perhaps is that the source term for K_z is negative through much of mid troposphere. However Smagorinsky points out that this was calculated as a small residual of two larger quantities and so the accurary of the computation is low.

We turn now to the problem of actually computing the atmospheric dissipation. The earliest such estimate was due to Brunt (1920) . Using typical cross-isobar flow in the friction layer and climatological wind maps he estimated the boundary layer dissipation to be $3Wm^{-2}$. He then estimated the dissipation in the free atmosphere as roughly the same and finally gave an estimate of $5Wm^{-2}$. Oort (1964) collated the then available data, most of which had calculated the dissipation as a residual and gave $2.3\ Wm^{-2}$ as the most reasonable value.

Fig. 5 shows his results separately for the space domain and the mixed space-time domain.

Holopainen (1963) and Kung (1966a,1966b,1967,1969) also provided estimates of the dissipation based on the evaluation of cross-isobar flow ($\underline{u}\cdot\nabla\phi$). Holopainen used data from eight stations in the U.K. for a few weeks in winter. Kung (1969) used data from a 5 year period over the fairly dense North American network.

The kinetic energy equation may be written

$$\frac{d}{dt}\underline{u}^2/2 = -\underline{u}\cdot\nabla\phi + \underline{u}\cdot\underline{F} \qquad 8$$

$\underline{U}.\underline{F}$ may be expected to be positive everywhere,when vertically integrated. $\underline{U}.\nabla\phi$ may vary widely and change sign. Thus an average for a limited area would be unlikely to be representative for the global region. Hence these workers also calculated the area and time average of $\frac{d}{dt}\underline{u}^2/2$ and calculated $\underline{U}.\underline{F}$ as a residual.

The calculation of the terms $-\underline{u}\cdot\nabla\phi$ and $\frac{d}{dt}\underline{u}^2/2$ is quite difficult, particularly the former. This is because the wind is approximately geostrophic and so is nearly perpendicular to $\nabla\phi$ .The technique used by Kung was to take all pairs of stations adjacent to a given station to get a set of estimates for $\nabla\phi$ These were then used to get a best estimate in a least square sense. The main problem in evaluating $\frac{d}{dt}\frac{1}{2}\underline{u}^2$ was to accurately estimate the advection of kinetic energy across the continental boundaries. Figure 6 shows the distribution of stations used in Kung's studies.

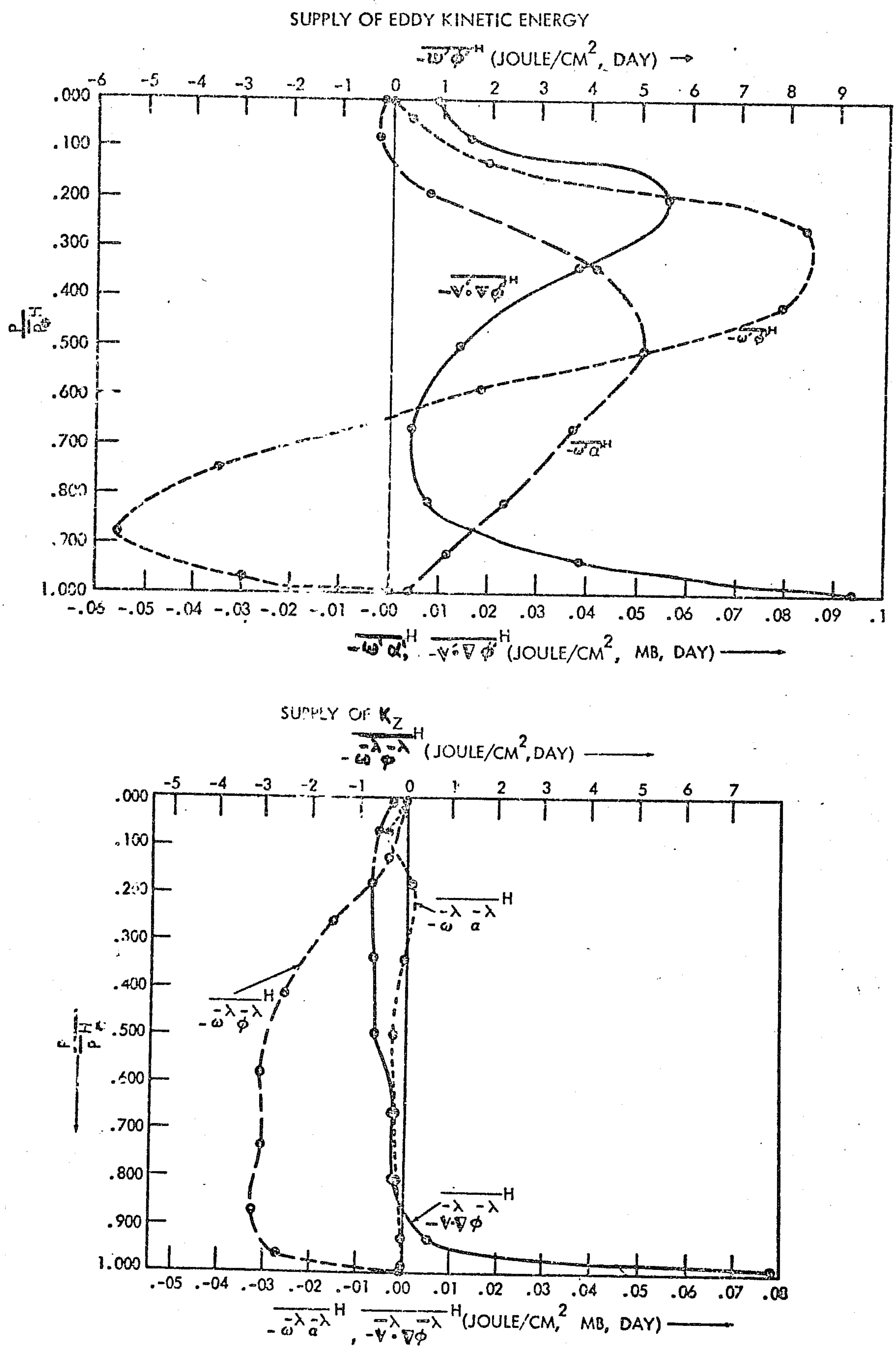
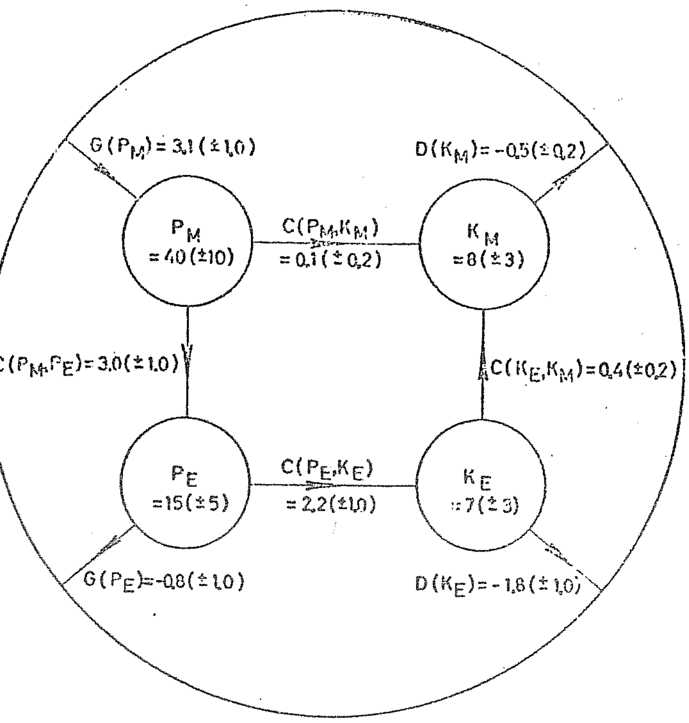


Figure 4:—The vertical distributions of $-\overline{V'\cdot\nabla\phi'}^H$, $-\overline{\omega'\alpha'}^H$, and $-\overline{\omega'\phi'}^H$ are shown in the upper part of the figure and those of $-\overline{\bar{V}^\lambda\cdot\nabla\bar{\phi}^\lambda}^H$, $-\overline{\bar{\omega}^\lambda\bar{\alpha}^\lambda}^H$, and $-\overline{\bar{\omega}^\lambda\bar{\phi}^\lambda}^H$ are shown in the lower part of the figure. These terms are on isobaric surfaces.



g. 5a: Tentative flow diagram of the atmospheric energy in the space domain. Values are averages over a year for the Northern Hemisphere. Energy units are in 10^5 joule m.$^{-2}$ (=10^5 erg cm.$^{-2}$); energy transformation units are in watt m.$^{-2}$ (=10^3 erg cm.$^{-2}$ sec.$^{-1}$).

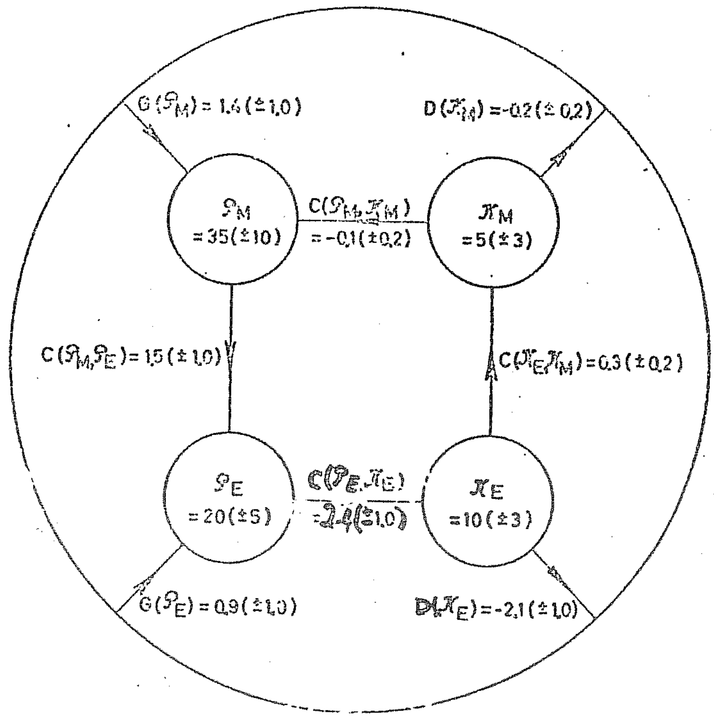


Fig. 5b—Tentative flow diagram of the atmospheric energy in the mixed space-time domain. Values are averages over a year for the Northern Hemisphere. Energy units are in 10^5 joule m.$^{-2}$; energy transformation units are in watt m $^{-2}$

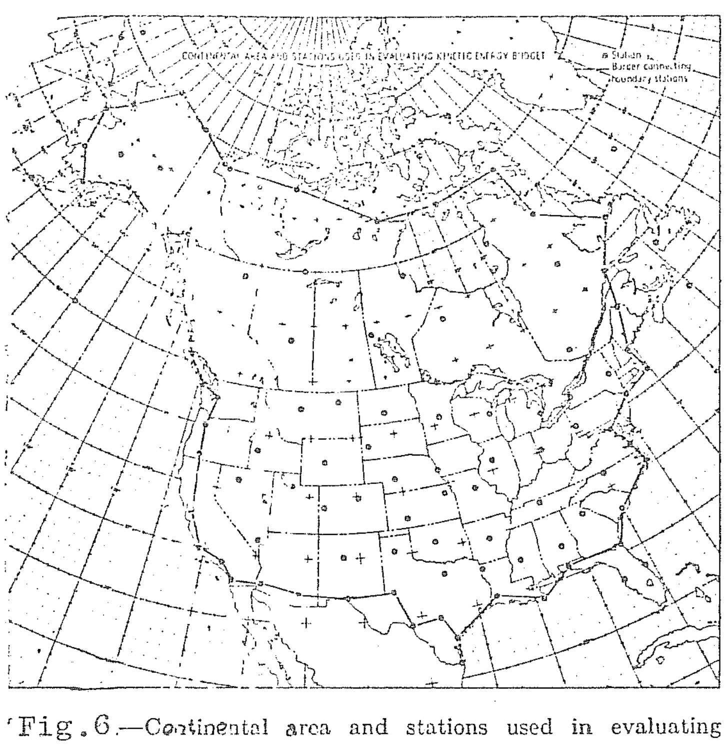


Fig. 6.—Continental area and stations used in evaluating kinetic energy budget.

Figures 7 to 9, taken from Kung (1967) show vertical profiles of the energy source and dissipation terms. Figure 7 shows the Winter, Summer and Annual Averages of $-\mathbf{V}\cdot\nabla\phi$ averaged over the continental area for a five year period. Figure 8 shows the vertical profile of kinetic energy dissipation for the same data.

We see that the source and sink terms are both large near the boundary layer and also near the tropopause, just as was found in the model results. An important feature of the results which does not directly concern us here, but has still to receive a satisfactory explanation, is shown in Fig. 9. This shows a marked difference in the results for 00Z and 12Z, roughly 1800 and 0600 local time. The strong diurnal variation in the boundary layer is in line with expectations but the variation near the tropopause is quite unexpected and there is controversy about the reality of the effect.

Taking the figures as they stand, the figures that result for the five year mean of the energy budget, using all the data is

$\bar{K}$	$\frac{\partial}{\partial t}\bar{K}$	$\frac{1}{A}\oint K\mathbf{V}\cdot\mathbf{n}\,dA$	$\frac{\partial}{\partial p}\overline{\omega K}$	$-\overline{\mathbf{V}\cdot\nabla\phi}$	$\overline{\mathbf{V}\cdot\mathbf{F}}$
16.60	-0.1	3.37	-.17	7.31	4.12

where $\bar{K}$ is in 10^5 joules/m^2 and the other quantities are in Wm^{-2}.

The value of 4.12 Wm^{-2} might be tentatively taken as the hemisphere value of dissipation. Of this dissipation, 50% took place in the lowest 100 mb, and 50% above this level. Kung argued that his values were likely to be underestimates of the dissipation. Nevertheless the estimates were almost double the value of 2.3Wm^{-2} given by Oort (1964) on the basis of results from a number of different authors.

Support for Kung's results may be found in the work of Ellsaesser (1969) and Trout and Panofsky (1969). Ellsaesser's estimates of turbulent dissipation used results from the Kolmogorov (1941) theory of turbulence to relate atmospheric wind variance statistics to the dissipation rate. Maps of wind variance for the Northern Hemisphere were prepared by Crutcher (1959-1962).

The expression used by Ellsaesser for the dissipation rate was

$$\varepsilon = \left[3\sigma^2(1-r_t)/17\right]^{3/2}/\bar{s}t \qquad 9$$

where σ is the vector standard deviation of the wind, r is the vector Eulerian time lag correlation, taken to be 0.793 for t = 6 hours, and $\bar{s}$ is the scalar mean wind. Ellsaesser evaluated this expression using Crutcher's maps at seven levels from 50 mb to 850 mb. Figure 10 shows the vertical integral of these results in units of Wm^{-2}. We notice that over the bulk of mid latitudes there is a typical value of 2 Watts M^{-2} with a maximum over the Atlantic of 3 Watts M^{-2}. Ellsaesser argues that because of his method of computation these values are underestimates by as much as a factor of 2 perhaps.

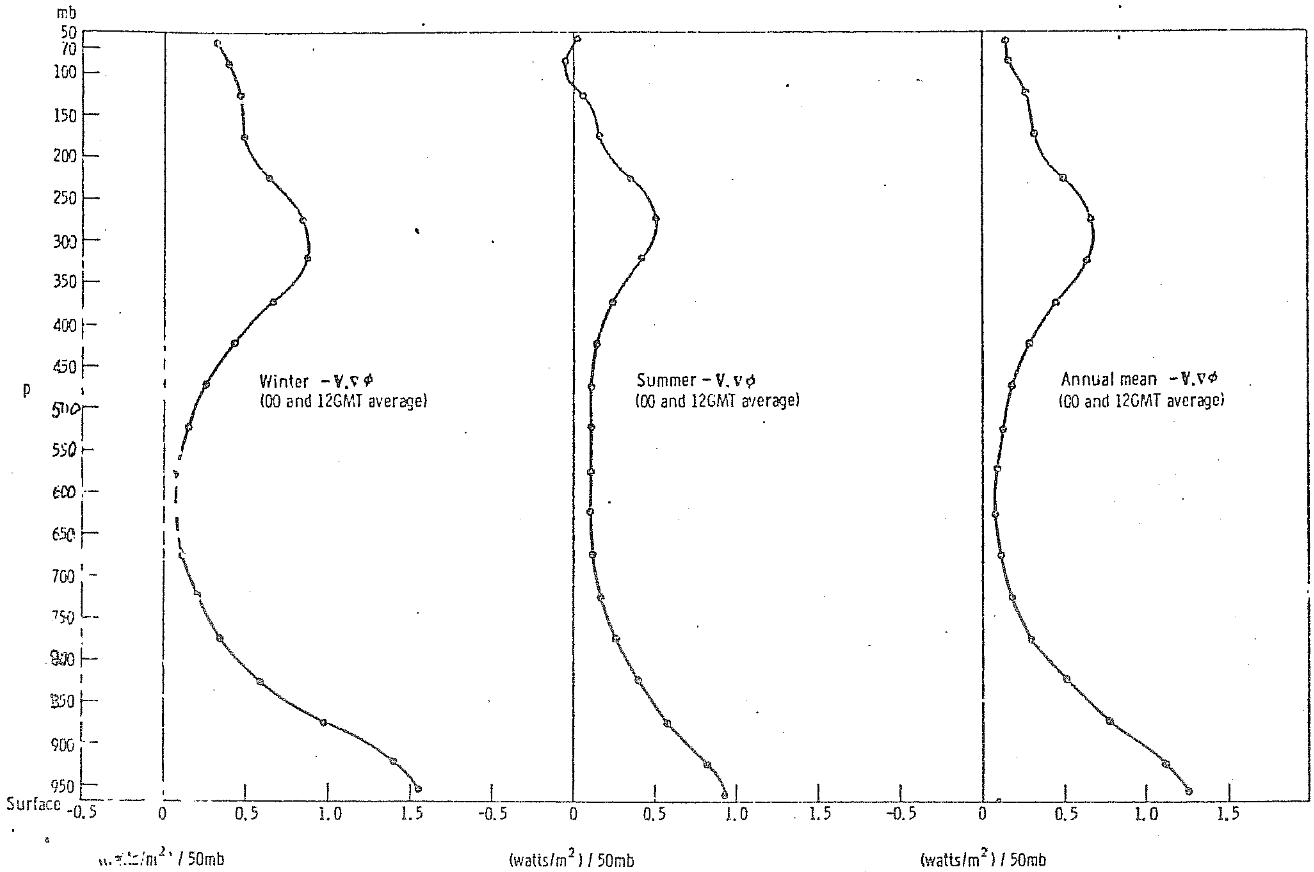


FIGURE 7—Vertical profile of the kinetic energy generation $-\overline{\mathbf{V}\cdot\nabla\phi}$ for the 5-yr. period (average of 00 and 12 GMT).

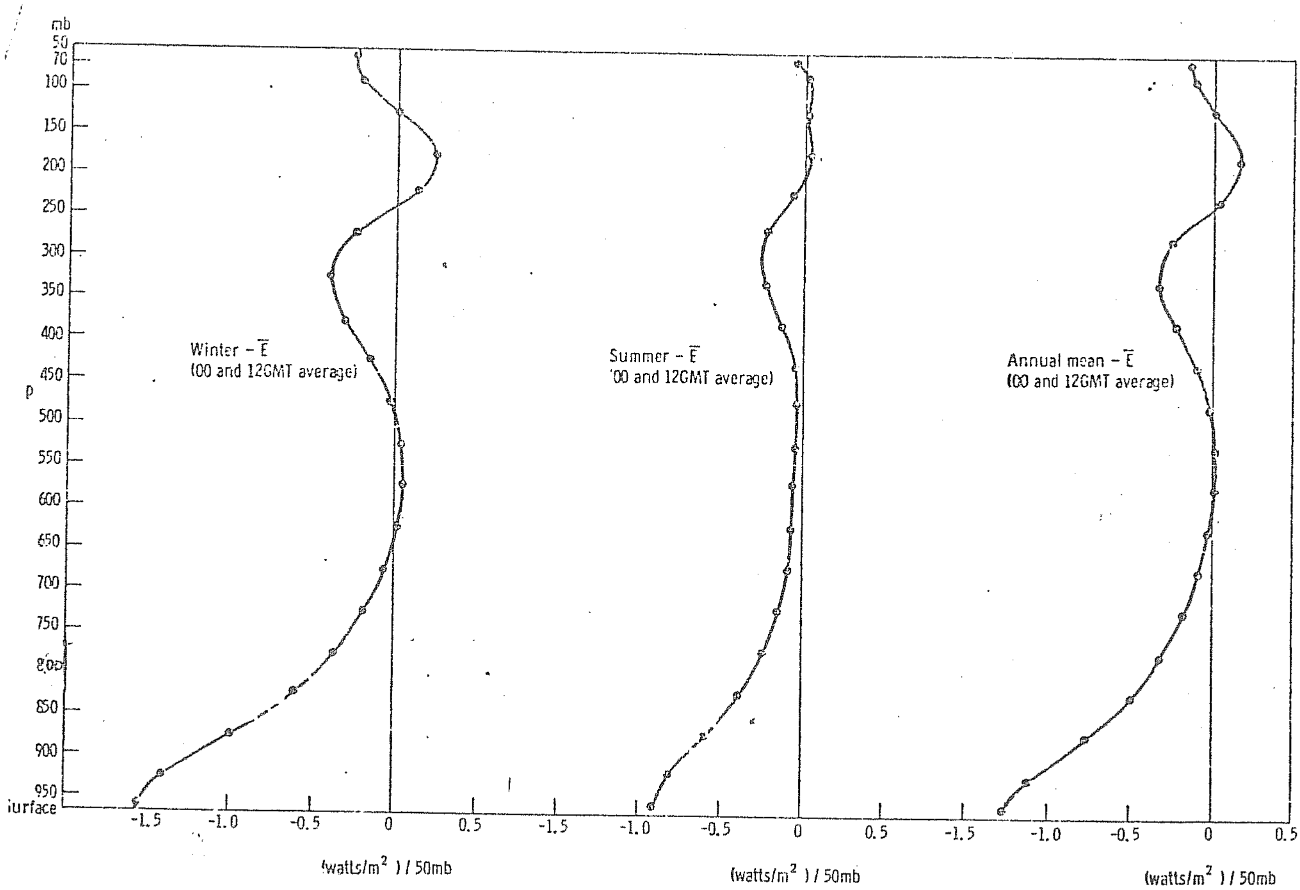


FIGURE 8—Vertical profile of the kinetic energy dissipation $\overline{E}$ for the 5-yr. period (average of 00 and 12 GMT).

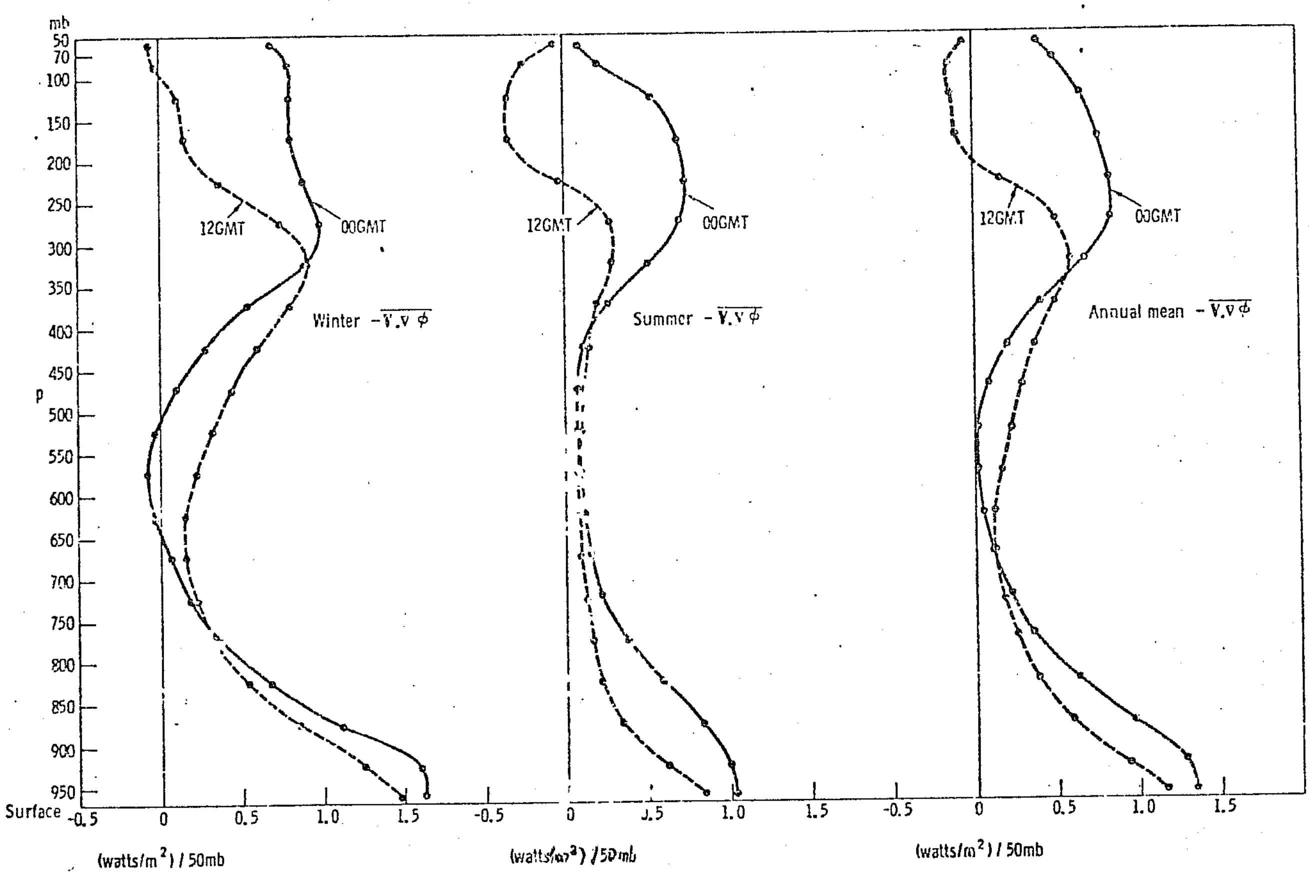


FIGURE 9a-Vertical profile of the kinetic energy generation $-\overline{\mathbf{V}\cdot\nabla\phi}$ at 00 and 12 GMT for the 5-yr. period.

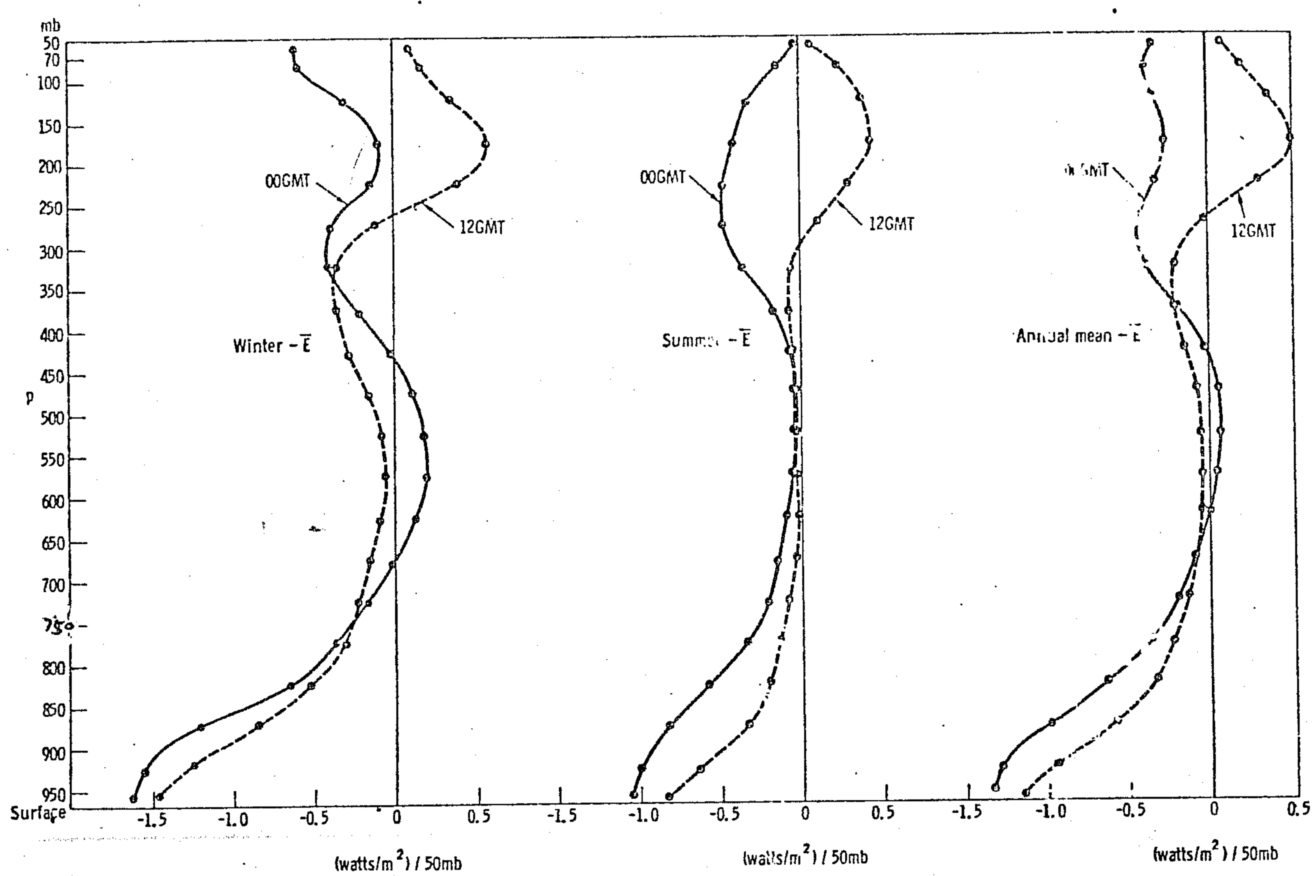


FIGURE 9b-Vertical profile of the kinetic energy dissipation $\overline{E}$ at 00 and 12 GMT for the 5-yr. period.

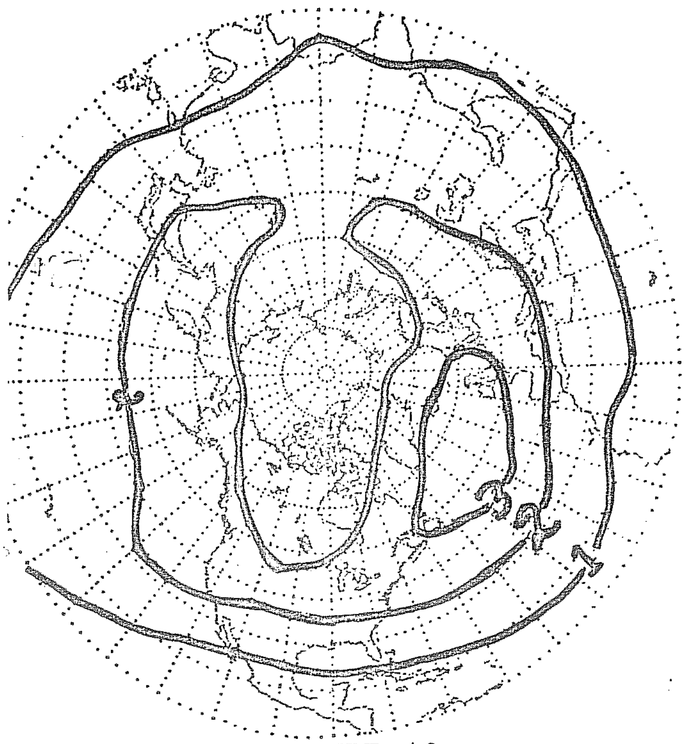


FIGURE 10

.—Contour analysis of E_f (total dissipation in the free atmosphere) computed as a 0- to 900-mb pressure integral of figures 1–7. Units are watts/m².

Trout and Panofsky (1969) used yet another data source to estimate the dissipation rate near the jet stream level. They use collections of atmospheric spectra made in different intensities (light, moderate, severe) of clear air turbulence to estimate the dissipation rate in these turbulence conditions from the Kolmagorov law

They further used statistics of the frequency of occurrence and intensity of clear air turbulence to construct estimates of the dissipation rate. Table 2 compares their results with Kung's.

TABLE 2

Layer feet x 10^3	Flight Data	Kung
25-30	0.59	0.59
30-34	0.36	0.37
34-40	0.37	0.32
TOTAL :	1.32	1.28

Dissipation Rates (Wm^{-2}) calculated by Trout and Panofsky (1969) and Kung (1967)

The agreement is remarkably close. Trout and Panofsky point out that the agreement is likely to be fortuitous as there is a substantial degree of uncertainty in the calculations. Nevertheless they lend support to Kung's finding.

Wiin-Nielsen (1968) has discussed the limitations of Kung's method of calculation. He makes the important point that the method leads to a value for $\nabla\phi$ which is appropriate for large scales while the winds used in the calculation do not have this feature. Newell et al (1971) and Newell et al (1974) discuss the energy budget in detail. Their method is quite different from Kung's and covers the hemisphere. Their calculations lead to an estimate of 2.2 Wm^{-2} for the dissipation rate.

Holopainen (personal communication) indicates that the extrapolation of Kung's results from North America to the hemisphere is probably not valid, based on calculations for other geographical areas.

The calculations by Newell and his collaborators are probably the most reliable available. It is an indication of the difficulty of these calculations that the work of Kung and others give a value for the space and time averaged atmospheric dissipation which is larger by a factor of two. Elsaesser's results for the free atmosphere give results which are as large as those of Newell and his collaborators for the whole atmosphere including the boundary layer.

Nevertheless, there is a consensus that dissipation away from the boundary layer and the tropopause region is relatively small. The bulk of the dissipation takes place near the surface in the boundary layer. Finally there is a body of evidence to indicate that dissipation near the tropopause may be as much as 25% of the total.

2. The differences between two and three dimensional turbulence

Much of the work done on the analysis and description of turbulence rests on the principles of dimensional analysis (Bridgman, 1931). Suppose we have a functional relationship between certain measured quantities and certain dimensional constants. Such an equation is called complete if it remains true formally without any change in the form of the function when the size of the fundamental units (of mass, length, time) is changed in any arbitrary fashion. It can be shown that if $\phi(\alpha, \alpha_2, \alpha_3 \ldots \alpha_n) = 0$ is a complete equation with n arguments and involves m fundamental units, then the equation can be written in the form

$$F(\pi_1, \pi_2, \ldots \pi_{n-m}) = 0 \qquad 1$$

where the π's are the n-m independent products of the arguments which are dimensionless in the fundamental units. This implicit relation may then be solved for say α_1, to give

$$\alpha_1 = \alpha_2^{x_2} \alpha_3^{x_3} \ldots \alpha_n^{x_n} \Phi(\pi_2, \pi_3, \ldots \pi_{n-m}) \qquad 2$$

where the x_i are such that $\alpha_1(\alpha_2)^{x2}(\alpha_3)^{x3} \ldots$ is dimensionless. This theorem has wide applicability in many areas of science.

In this talk we are going to consider some of the differences between 3-dimensional and 2-dimensional turbulence. In the next talk we discuss the relevance of 2-dimensional turbulence theory to atmospheric flow.

If we use upper case letters U, S etc. to denote mean flow quantities and lower case letters u, s, for fluctuating quantities, then the equations of motion for steady mean flow in an incompressible fluid are (Tennekes and Lumley, 1972)

$$U_j \frac{\partial}{\partial x_j} U_i = \frac{\partial}{\partial x_j} \frac{1}{\rho} T_{ij} \qquad 3$$

$$\frac{\partial}{\partial x_j} U_j = 0 \qquad 4 \qquad (1)$$

where the stress tensor T_{ij} is

$$T_{ij} = -P\delta_{ij} + 2\mu S_{ij} - \rho \overline{u_i u_j} \qquad 5$$

where P is the pressure, ρ is the density, μ the dynamic viscosity and $(\overline{\ })$ is a horizontal average.

The mean rate of strain S_{ij} is defined by

$$S_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}U_i + \frac{\partial}{\partial x_i}U_j\right) \qquad 6$$

In studying turbulent flows the energy and vorticity budgets are of central importance.

If we multiply (1) by U_i we have

$$U_j \frac{\partial}{\partial x_j}\frac{1}{2}U_iU_i = \frac{\partial}{\partial x_j}(T_{ij}U_i) - T_{ij}S_{ij} \qquad 7$$

since

$$T_{ij}\frac{\partial}{\partial x_j}U_i = T_{ij}S_{ij}$$

as T and S are symmetric.

The first term on the right is a transport term which serves only to redistribute the kinetic energy of the mean flow. The second term is called the deformation work term and represents a loss or gain of mean flow kinetic energy to the turbulence. When we expend T_{ij} this term becomes

$$T_{ij}S_{ij} = -P\delta_{ij}S_{ij} + 2\mu S_{ij}S_{ij} - \rho\overline{u_iu_j}S_{ij} \qquad 8$$

$$= 2\mu S_{ij}S_{ij} - \rho\overline{u_iu_j}S_{ij}$$

The viscous term, $2\mu S_{ij}S_{ij}$, in the deformation work is positive definite and always represents a loss of mean flow kinetic energy. Note that it is related to the symmetric part of $\frac{\partial}{\partial x_j}U_i$, the strain rate, rather than to the skew-symmetric part, the vorticity.

The Reynold's stress term $\rho\overline{u_iu_j}S_{ij}$ tends to be dissipative in most flows. The loss of mean flow kinetic energy will show up as a gain of turbulent energy and so this term is known as the turbulent energy production.

In most typical circumstances the viscous terms in the mean flow kinetic energy equation are smaller by $O(R_e) = u\ell/\nu$ than the turbulence terms. Here u is a typical velocity scale, ν is the kinematic viscosity and ℓ an integral length scale. The turbulent energy budget can be derived in a straight forward way and reads

$$U_j \frac{\partial}{\partial x_j} \frac{1}{2} \overline{u_i u_i} = -\frac{\partial}{\partial x_j}\left(\frac{1}{\rho} \overline{p u_i} + \frac{1}{2} \overline{u_i u_i u_j} - 2\nu \overline{u_j s_{ij}}\right) - \overline{u_i u_j} S_{ij} - 2\nu \overline{s_{ij} s_{ij}} \qquad 9$$

where $s_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

In a steady homogeneous pure shear flow where all averaged quantities except U_i are independent of position the above reduces to

$$-\overline{u_i u_j} S_{ij} = 2\nu \overline{s_{ij} s_{ij}} \qquad 10$$

Using the scaling

$$\overline{u_i u_j} \sim u^2 , \quad S_{ij} \sim u/\ell$$

we have

$$C_1 \frac{u\ell}{\nu} S_{ij} S_{ij} = \overline{s_{ij} s_{ij}} \qquad 11$$

where $C_1 = 0(1)$

Since $\frac{u\ell}{\nu} = R_e >> 1$ we have

$$\overline{s_i s_j} >> S_{ij} S_{ij}$$

Thus the fluctuating rate of strain is very much larger than the mean rate of strain at large Reynolds number. Moreover since S has the dimensions of $(\text{time})^{-1}$ this implies that the bulk of the energy dissipation takes place on very short time and space scales. This suggests that there should be very little direct interaction between the strain rate fluctuations which are mostly involved in the dissipation, and the mean flow.

The energy exchange between the mean flow and the turbulence is governed by the dynamics of the large eddies; the turbulent production is proportional to eddy size. The energy extracted by the turbulence from the mean flow enters the turbulence mainly at scales comparable to the integral scale

$$\ell = \left(\int_0^\infty u(x_2) u(0)\, dx_2\right) / \overline{u_i u_j} \qquad 12$$

Viscous dissipation takes place at scales comparable to the Kolmogorov microscale $\eta = (\nu^3/\varepsilon)^{1/4}$ where ε is the dissipation rate. The internal dynamics of the turbulence must then transfer energy from large scales to small scales.

If $\tilde{\omega}_i$ represents the total vorticity vector then the vorticity equation may be written in the form (Tennekes and Lumley 1972)

$$\frac{\partial}{\partial t}\tilde{\omega}_i + \tilde{u}_j \frac{\partial}{\partial x_j}\tilde{\omega}_i = \tilde{\omega}_j \tilde{s}_{ij} + \nu \frac{\partial^2 \tilde{\omega}_i}{\partial x_j \partial x_j} \qquad 13$$

The term $\tilde{\omega}_j \tilde{s}_{ij}$ represents a rotation and an amplification of the vorticity. It can be interpreted as an expression of the conservation of angular momentum. Suppose a fluid element has vorticity $\tilde{\omega}$ „ and is being stretched along the x_1 axis, $\tilde{s}_{11} > 0$, so that its x_1 moment of inertia is decreasing. Then the rotation rate about the x_1 axis $\tilde{w}_1$ must increase to conserve angular momentum. This term involves changes of scale of the flow and is responsible for the transfer of energy from large scales to small scales in three dimensional flow. This term is identically zero in two-dimensional flow and this fact is the reason for the crucial differences between two dimensional and three dimensional turbulence.

With $\tilde{\omega}_i = \Omega_i + \omega_i$, $\overline{\omega_i} = 0$, the squared vorticity budgets for steady flow are:

$$U_j \frac{\partial}{\partial x_j}\frac{1}{2}\Omega_i \Omega_i = -\frac{\partial}{\partial x_j}\left(\Omega_i \overline{\omega_i u_j}\right) + \overline{u_j \omega_i}\frac{\partial}{\partial x_j}\Omega_i + \Omega_i \Omega_j S_{ij} + \Omega_j \overline{\omega_j s_{ij}} + \nu \frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{1}{2}\Omega_i \Omega_i\right) - \nu \frac{\partial \Omega_j}{\partial x_j}\frac{\partial \Omega_j}{\partial x_j} \qquad 14$$

$$U_j \frac{\partial}{\partial x_j}\frac{1}{2}\overline{\omega_i \omega_i} = -\overline{u_j \omega_i}\frac{\partial \Omega_i}{\partial x_j} - \frac{1}{2}\frac{\partial}{\partial x_j}\left(\overline{u_j \omega_i \omega_i}\right) + \overline{\omega_i \omega_j s_{ij}} + \overline{\omega_i \omega_j}S_{ij} + \Omega_j \overline{\omega_i s_{ij}} + \nu \frac{\partial^2}{\partial x_j \partial x_j}\frac{1}{2}\overline{\omega_i \omega_i} - \nu \frac{\partial \omega_j}{\partial x_j}\frac{\partial \omega_j}{\partial x_j} \qquad 15$$

In two dimensions the turbulent energy and enstrophy budgets are

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{u_i u_i} = -\frac{\partial}{\partial x_j}\left(\overline{u_j u_i u_i/2}\right) - \frac{\partial}{\partial x_i}\left(\frac{1}{\rho}\overline{u_i p}\right) + \nu \frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{1}{2}\overline{u_i u_i}\right) - \nu \left(\frac{\partial u_j}{\partial x_j}\right)^2 \qquad 16$$

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{\omega_i \omega_i} = -\frac{\partial}{\partial x_j}\left(\overline{u_j \omega_i \omega_i/2}\right) + \nu \frac{\partial^2}{\partial x_j \partial x_j}\left(\frac{1}{2}\omega_i \omega_i\right) - \nu \left(\frac{\partial \omega_j}{\partial x_j}\right)^2 \qquad 17$$

Letting $\bar{\bar{E}} = \int E(k)dk$ we define (Lilly, 1973)

$$(k_2)^2 = \frac{\frac{1}{2}\overline{\omega^2}}{\bar{E}} = \frac{\int k^2 E(k)\,dk}{\bar{E}}$$

$$(k_4)^4 = \frac{\overline{(\nabla\omega)^2}}{\bar{E}} = \frac{\int k^4 E(k)\,dk}{\bar{E}}$$

It may be shown that

$$\frac{d}{dt}\ln\bar{E} = -\nu k_2^2 \qquad 18$$

$$\frac{d}{dt}\ln\tfrac{1}{2}\overline{\omega^2} = -\nu\, k_4^4/k_2^2 \qquad 19$$

Now $k_4 >> k_2$ if E(k) is reasonably broad (e.g. a power-law spectrum); hence dissipation of enstrophy is much greater than dissipation of energy in two dimensional turbulence.

The diffusion approximation for turbulence (Leith 1968, 1969) provides a heuristic derivation of the k^{-3} spectrum of two-simensional turbulence.

We define the mean kinetic energy and the mean enstrophy per unit mass as

$$\tfrac{1}{2}\overline{u^2} = \int_0^\infty E(k)\,dk$$

$$\tfrac{1}{2}\overline{\omega^2} = \int_0^\infty G(k)\,dk$$

where $G = k^2 E$

The spectral form for the energy and enstrophy budgets may be written

$$\frac{\partial E}{\partial t} = I - 2\nu k^2 E \qquad 20$$

$$\frac{\partial G}{\partial t} = J - 2\nu k^2 G \qquad 21$$

where I,J are inertial transfer terms with the property that $\int_0^\infty I\,dk = 0$ and $\int_0^\infty J\,dk = 0$. Thus we may postulate that I,J may be written as $I = -\frac{\partial F}{\partial k}$, $J = -\frac{\partial H}{\partial k}$ where F,H are the fluxes of E,G through wave number space.

We require that $\frac{\partial H}{\partial k} = k^2\frac{\partial F}{\partial k}$ to preserve the relation $G = k^2 E$ This implies the following restriction on F which will prove useful

$$0 = \int_0^\infty \frac{\partial H}{\partial k}dk = \int_0^\infty k^2\frac{\partial F}{\partial k}dk = k^2 F\Big|_0^\infty - 2\int_0^\infty kF\,dk = -2\int_0^\infty kF\,dk$$

$$\therefore \int_0^\infty kF\,dk = 0 \qquad 22$$

Suppose we write $F = D\,\frac{\partial Q}{\partial k}$ so that I now has the form of a diffusion. Then since E has the dimensions $(\text{length})^3(\text{time})^{-2}$ the combination DQ has the dimensions $(\text{length})(\text{time})^{-3}$.

Suppose we seek a similarity relationship $DQ=\beta_2 k^m E^n$ then we find $m=7/2$ and $n=3/2$ and finally $F=\beta_2 k^{7/2-\gamma}\frac{\partial}{\partial k}k^{-\gamma}E^{3/2}$ where γ is still to be determined.

The parameter γ is determined by requiring $\int_0^\infty kF\,dk=0$

i.e.

$$\int_0^\infty k^{9/2-\gamma}\frac{\partial}{\partial k}k^{\gamma}E^{3/2}=0$$

Therefore $\gamma=9/2$ and $F=-\beta_2 k^{-1}\frac{\partial}{\partial k}k^{9/2}E^{3/2}$ 23

It may be verified that this form for F implies, through the relation $\frac{\partial H}{\partial k}=k^2\frac{\partial F}{\partial k}$, that H is of the form

$$H=-\beta_2 k^3\frac{\partial}{\partial k}k^{-1/2}G^{3/2} \qquad 24$$

F and H are the energy and enstrophy cascade rates.

If E has the Kolmogorov distribution $E = A k^{-5/3}$ then H=0 and F is constant. If E has the distribution $E=Ak^{-3}$ then $F\equiv0$, and H is constant and has opposite sign to the sign of F in a $k^{-5/3}$ spectrum. Thus two equilibrium ranges can exist in two-dimensional turbulence. The k^{-3} spectrum will extend to the highest wave numbers. It will cascade enstrophy to higher wave numbers but the energy cascade will be negligible. The $k^{-5/3}$ range, if it exists, will occur at the low wave number end of the spectrum and will transfer energy to the longest waves.

The reasons for these phenomena are best seen by considering the results of Fjörtoft (1953). Fjörtoft showed first that an energy exchange between just two modes is impossible.

Let ΔE_r, ΔE_s be the energy changes in these modes, $r\neq s$

Then the conservation laws for energy and enstrophy give

$$\Delta E_r + \Delta E_s = 0$$

$$r^2\Delta E_r + s^2\Delta E_s = 0$$

The only solution is $\Delta E_r = \Delta E_s = 0$

Suppose then that we have three modes $s > r > p$

Then $\Delta E_p + \Delta E_r + \Delta E_s = 0$

$$p^2 \Delta Ep + r^2 \Delta E_r + s^2 \Delta E_s = 0$$

$$\Delta Ep = - \frac{s^2-p^2}{s^2-p^2} \Delta E_r \qquad 25$$

$$\Delta E_s = - \frac{p^2-p^2}{s^2-p^2} \Delta E_r \qquad 26$$

Thus a loss of energy in the intermediate wave number is balanced by a gain at the shorter and longer wave numbers. The question as to whether the longest wave or the shortest wave receives most energy has been studied by Wiin-Nielsen (1974) and Merrilees and Warn (1975). The latter authors showed that in ~70% of possible triad interactions more energy went to the largest wavelength while in ~60% of possible triad interactions, more enstrophy goes to the shortest wavelength.

These ideas have been tested in numerical integrations by Lilly (1971). Figs. 1-4 are taken from Lilly (1971) showing an experiment with forcing at wave number 8. Fig. 1 shows the stream function, Fig. 2 the vorticity and Fig. 3 the forcing function at a time when the turbulence is fully developed. Fig. 4 shows the energy spectrum at the same time. We see that most of the energy is in the longest waves with an approximately k^{-3} tail to the spectrum. Although the forcing is at wave number 8, the bulk of the energy has accumulated in the longest waves.

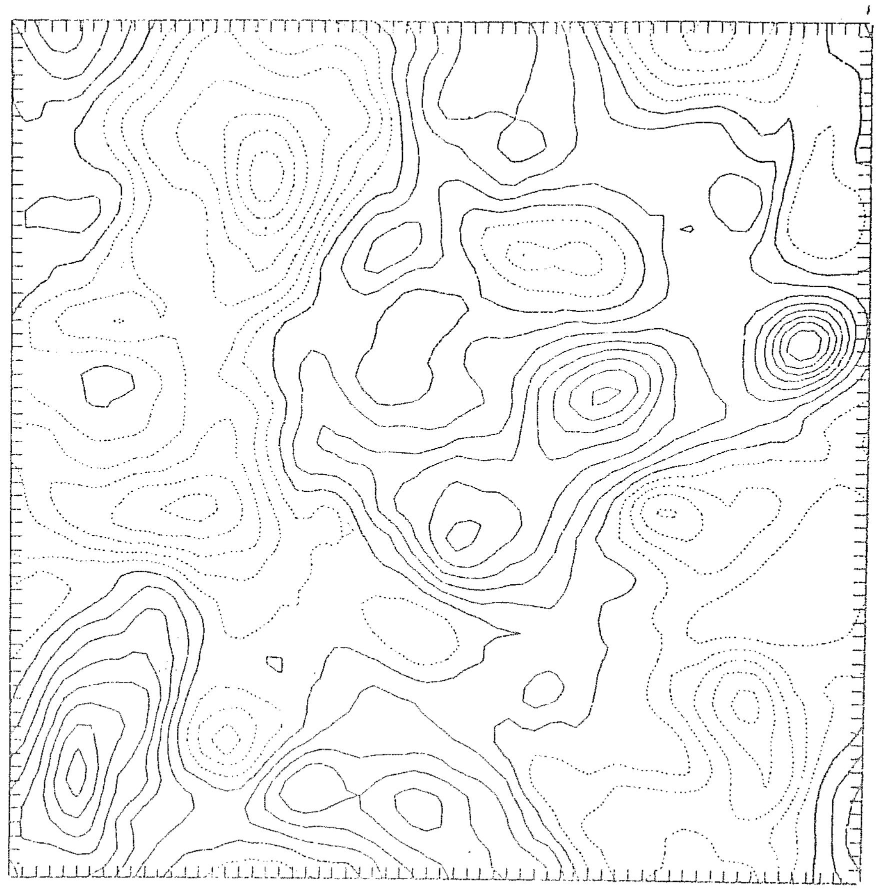


Fig. 1 Stream function map developed at the 2200th time step (22.0 time units) of a numerical simulation of two-dimensional turbulence with surface drag. The mesh interval is shown by hatch marks along the boundaries. Positive contours are solid, negative dashed, with contour interval 0.095. The mesh interval is 0.125, $\nu = 5.0 \times 10^{-4}$, and $K = 0.02$. The forcing is of the negative viscosity type, i.e. $F = \omega_F/\tau$, where ω_F is the portion of the vorticity field associated with wave number 8. The time constant $\tau = 2.0$. All units are dimensionless. The isoline analysis was performed by the computer, using a two-dimensional linear interpolation scheme, and photographed from a cathode-ray-tube presentation. (from Leith 1973)

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Fig. 2 Vorticity map at the same time as Fig. 1 . Contour interval = 2.24. (from Leith 1973)

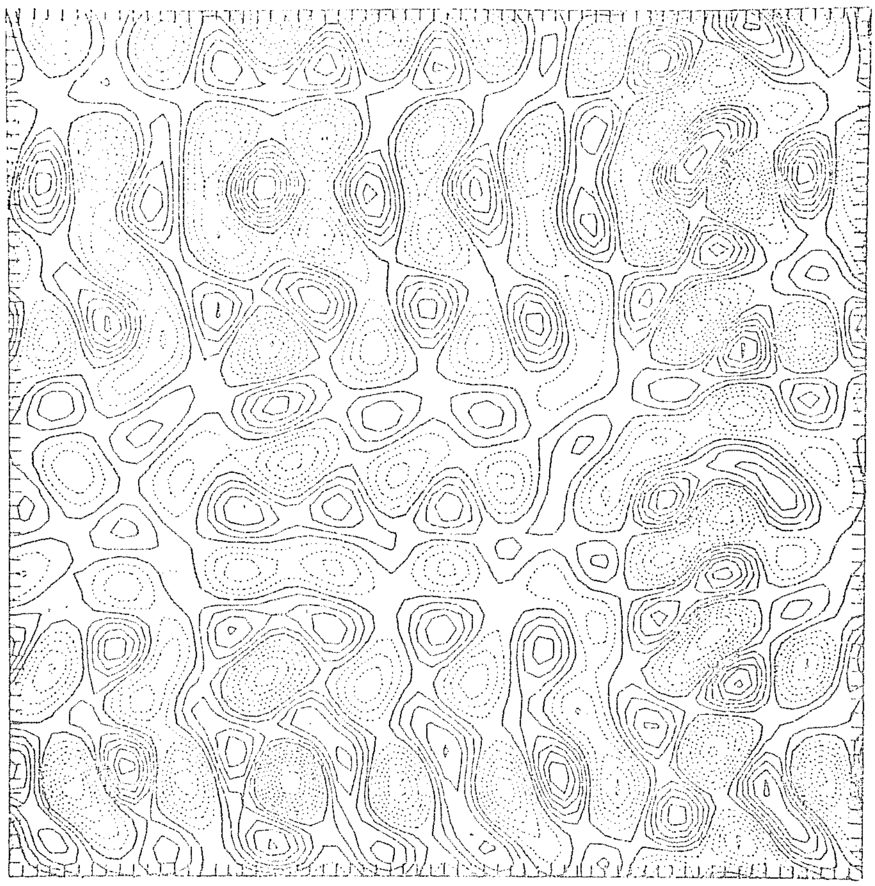

Fig. 3 Forcing function map at the same time as Fig. 5.1.
Contour interval = .196. (from Leith 1973)

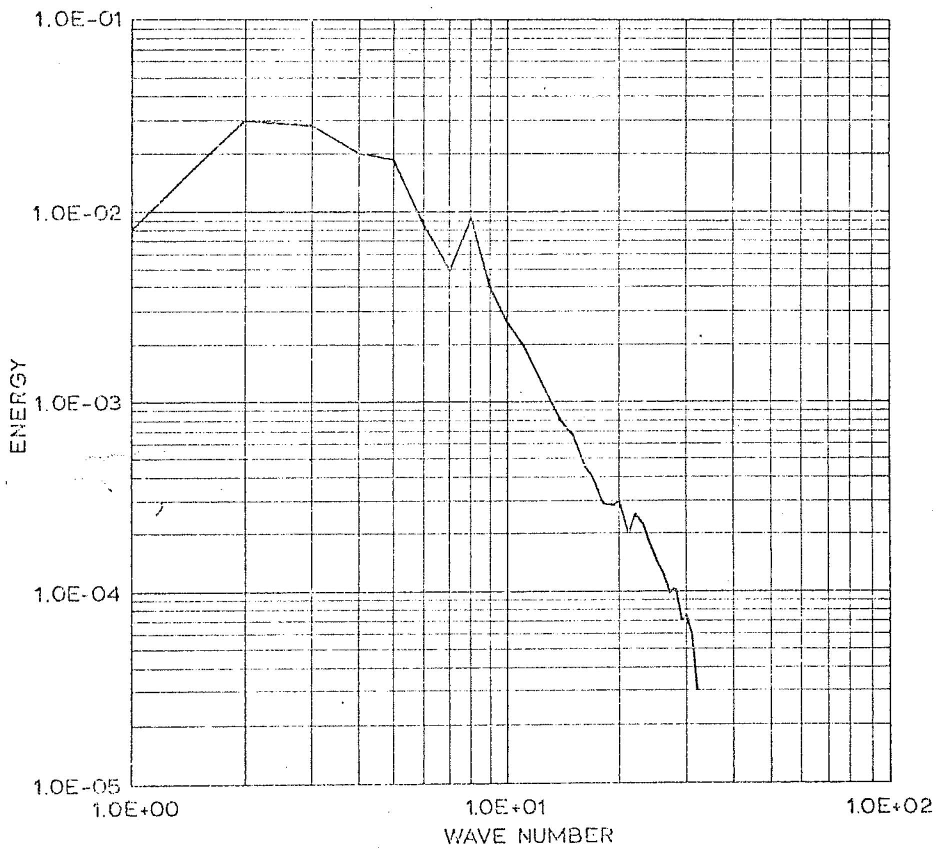

Fig. 4 The scalar energy spectrum for the same time as Fig. 5.1. The plotted values represent the energy associated with the surface of a square in wave number space rather than the circle normally assumed in turbulence theory. (from Leith 1973)

3. Geostrophic turbulence; Practical Closure Formulations

In this lecture we shall consider the relevance of the theory of quasi-geostrophic turbulance to two-dimensional turbulence; we shall consider the effects of various finite difference schemes on the representations of turbulences. Finally, we shall consider some of the parameterisations that have been proposed to represent the effects of unresolved scales of turbulence on resolved scales.

Many authors (e.g. Horn and Bryson, 1963; Wiin-Nielsen, 1967; Julian et al., 1970; Julian and Cline, 1974) have calculated atmospheric energy spectra from observations. Fig. 1, taken from Julian et al. (1970) is representative of the results. It shows clearly the fact that the atmospheric spectrum falls off between wave numbers 10-20 with a power law dependence that is close to k^{-3}. At about the time that these calculations were being made, Kraichnan (1967) showed that two dimensional turbulence has two inertial ranges, a $k^{-5/3}$ energy/cascading range and a k^{-3} enstrophy cascading range. The enstrophy range does not cascade energy and vice versa. Moreover, the directions of the transfers are opposite in the two ranges so that if the enstrophy cascade is to short wave-lengths, the energy (de)-cascades to long wave-lengths, and vice versa.

These properties of the two-dimensional turbulence were deduced from the two conservations laws of energy and squared vorticity (enstrophy). Charney (1971) pointed out that three dimensional quasi-geostrophic flow is governed similarly by the conservation of energy and of quasi-geostrophic potential vorticity.

A concise derivation of this latter conservation law is given by Charney and Stern (1961).

The quasi-geostrophic equations are

$$\left(\frac{\partial}{\partial t}+\underline{V}_\psi\cdot\nabla\right)\left(\nabla^2\psi+\beta\psi\right)+\left(f_0/\bar{\rho}\right)\left(\bar{\rho}\,w\right)_z=0 \qquad 1$$

$$\left(\frac{\partial}{\partial t}+\underline{V}_\psi\cdot\nabla\right)\frac{\partial\psi}{\partial z}+\frac{N^2}{f_0}w=0 \qquad 2$$

where ψ is the stream function, $\underline{V}_\psi$ is the lowest order term in the non-divergent wind, N is the buoyancy frequency and f_0 the Coriolis parameter. Using the non-dimensional continuity equation

$$\bar{\rho}\,\nabla\cdot\underline{V}_\sigma+\frac{\partial}{\partial z}\left(\bar{\rho}\,w\right)=0 \qquad 3$$

where $\bar{\rho}$ is the horizontal mean density and $\underline{V}_\sigma$ is the lowest order term in the divergent wind.

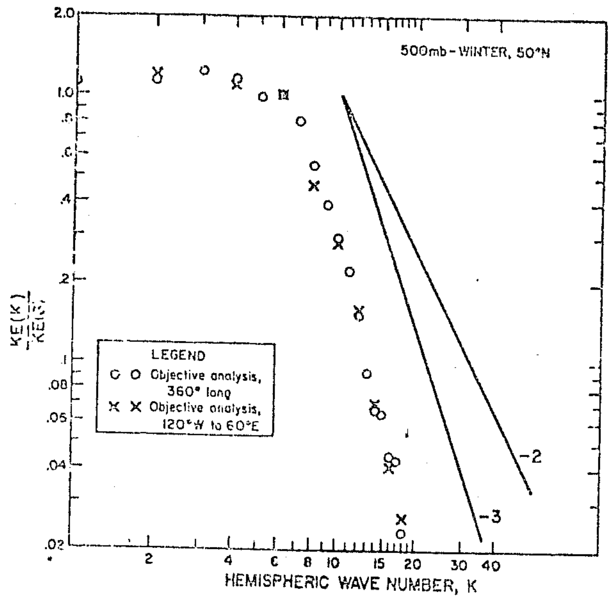

FIG. 1. Kinetic energy spectra, 500 mb, 50N, for the winter season (data sets Iw and IIw, Table 1) plotted on a full logarithmic scale. Shown are the spectral estimates using data from all longitudes and for a 180° segment from 120W to 60E. For purposes of comparison all estimates have been standardized by division by the estimate for wavenumber 6. Julian et al. (1970)

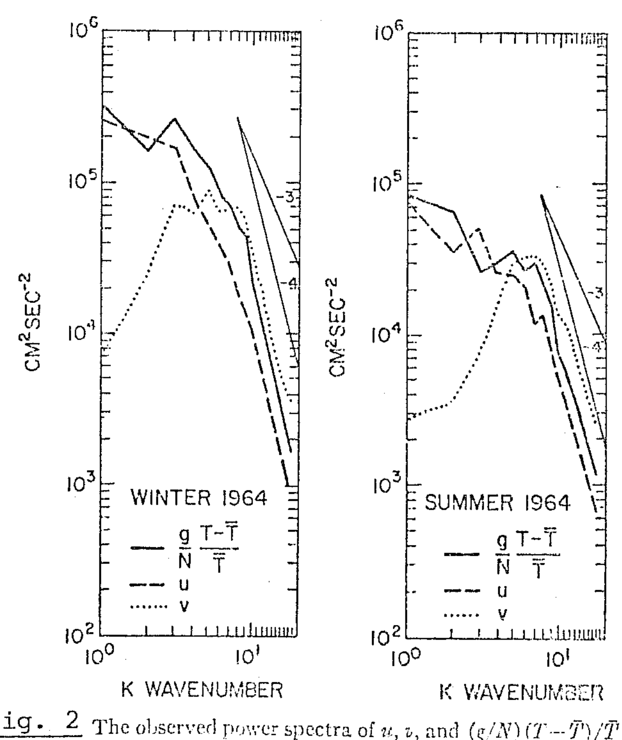

Fig. 2 The observed power spectra of u, v, and $(g/N)(T-\bar{T})/\bar{T}$ at 40N and 500 mb for winter and summer 1964 as a function of the hemispheric wavenumber $K=k_1$.

Now eliminate W and $\nabla . \underline{V}_0$ to give

$$\left(\frac{\partial}{\partial t} + \underline{V}_\psi . \nabla\right) \mathcal{L}(\psi) + \beta \psi_x = 0, \qquad 4$$

where $$\mathcal{L}(\psi) = \nabla^2 \psi + \frac{f_0^2}{\bar{\rho}} \left(\frac{\bar{\rho}}{N^2} \psi_z \right)_z$$

If we multiply this equation respectively by $\bar{\rho}\psi$ and $\bar{\rho}L(\psi)$ and integrate over space we get an energy equation and an equation for quasi-geostrophic potential enstrophy

$$\frac{dE}{dt} = \frac{d}{dt} \iiint \tfrac{1}{2} \left(\nabla\psi . \nabla\psi + \frac{f_0^2}{N^2} \psi_z^2 \right) \bar{\rho} \, dx\, dy\, dz = 0 \qquad 5.a$$

$$\frac{dF}{dt} = \frac{d}{dt} \iiint \mathcal{L}^2(\psi) \bar{\rho} \, dx\, dy\, dz = \beta \iint \bar{\rho} \left(\frac{f_0^2}{N^2} \psi_x \psi_z \right)_{z=0} dx\, dy \qquad (5.b$$

Let us suppose that $\psi_z = 0$ at $z = 0$, i.e. that the ground is an isothermal surface. Then

$$\frac{dF}{dt} = 0 \qquad 6$$

Suppose the flow is periodic in the x-direction and is bounded by walls at $y = y_1, y_2$

Then L is a self-adjoint elliptic operator with a complete orthonomal set of eigenfunctions ψ_m with eigenvalues λ_m satisfying the homogenous boundary conditions $(\psi_m) = 0$ at $z = 0$ and either $\bar{\rho} |\nabla \psi_m|^2 \to 0$ or $\bar{\rho} |\psi_m \psi_{mz}| \to 0$ as $z \to \infty$

Now we can order the λ_i so that $0 \leq \lambda_1 \leq \lambda_2 \ldots$ and so that $\lambda_m^{1/2} \sim m^{1/3}$ as $m \to \infty$

Moreover, the modal surfaces of these eigenfunctions divide the fluid into m subdomains of volume $\sim (m^{1/3})^3$

From completeness of the ψ_m we have

$$\psi = \sum_1^\infty a_m \psi_m$$

and we can show that

$$2E = \sum_1^\infty \lambda_m a_m^2 = \sum_1^\infty b_m = \text{constant} \qquad 7$$

$$2F = \sum_1^\infty \lambda_m^2 a_m^2 = \sum_1^\infty \lambda_m b_m = \text{constant} \qquad 8$$

Since the series for F majorises the series for E by the factor λ_m which tends to infinity like $m^{2/3}$ an energy cascade is impossible.

This is not true if the ground is not isentropic or isothermal. In such a situation, we can have the formation of frontal near-discontinuity and therefore an energy cascade.

Assuming that the flow in the interior is not substantially influenced by the presence of the boundaries and assuming that

1. the horizontal scale is small compared to $2\lambda = \frac{2HN}{f_0}$, the Rossby radius, and the vertical scale in the stretched coordinate $d\rho = N/f_0 \, dz$ is similarly small,

2. the excitation energies are so large that the β-term is small compared to the advection terms,

3. N varies more slowly in the vertical than does the turbulence intensity,

4. away from boundaries the turbulence is locally homogeneous and isotropic,

5. non-linear interactions are local in wave-number space,

6. the input energy is high enough to guarantee an inertial sub-range so that there is no internal viscous dissipation;

then the conservation law for quasi-geostrophic potential vorticity becomes

$$\chi_t = J(\nabla^2 \chi, \chi) \qquad 9$$

where $\chi = \frac{\bar{\rho}(z)}{\bar{\rho}(0)} \psi$, $\quad d\rho = \frac{N}{f_0} dz \quad$ and

$$\nabla^2 \chi = \chi_{xx} + \chi_{yy} + \chi_{\rho\rho}$$

Charney then presents arguments to support his contention that ($\chi_x, \chi_y, \chi_\rho$) is homogenous and isotropic in x, y, ρ Then, following Kraichnan the scalar energy spectrum E(k) depends only on the quasi-geostrophic potential vorticity dissipation function η. Then, by dimensional arguments,

$$E(k) = C \eta^{2/3} k^{-3} \quad \text{where } k^2 = k_1^2 + k_2^2 + k_3^2 \qquad 10$$

Because of isotropy, the energy spectrum for χ_x, χ_y, χ_z has the same form. Thus, there is equi-partition of energy among the x, y components of kinetic energy and the available potential energy.

Fig. 2 shows calculations of energy spectra from real-data Kao (1970). We notice the approximate equi-partition when the temperature departure $T - \bar{T}$ is scaled by $g/N\bar{T}$, and the approximate k^{-3} behaviour beyond wave number 8.

We note as a consequence of the theory that kinetic energy dissipation must take place mostly in the longest waves, in the boundary layer at the surface and near tropopause.

It should be clear from these results that the maintenance of conservation laws in numerical models is of crucial importance for the long term behaviour of the flow.

The adiabatic equations of motion have an exact conservation law due to Ertel

$$\frac{d}{dt}\left(\frac{2\underline{\Omega} + \nabla\times\underline{V}}{\rho}\cdot\nabla s\right) = 0 \qquad 11$$

where s is specific entropy. The quasi-geostrophic potential vorticity is an approximate form of this conserved quantity. In hydrostatic flow a similar law obtains except that $2\underline{\Omega}$ is replaced by its vertical component and $\nabla\times\underline{V}$ is computed as if the vertical velocity were zero.

It is possible to formulate a model which will conserve this quantity although such a model would be expensive because we would have to use iterative methods to extract either winds or temperatures from the non-linear potential vorticity.

The finite difference formulation to be used in the Centre's first forecast model will conserve energy and for horizontal non-divergent flow will conserve $p_s Z^2$ where Z is the absolute vorticity about the local vertial and p_s is the surface pressure.

Sadourny (1975) has considered the effects of incorporating the conservation laws in the finite difference scheme. He took two almost identical models for the shallow water equations on a (non-rotating) plane with doubly periodic boundary conditions. There were slight differences in the rotational terms of the equations such that one conserved energy and the other potential enstrophy.

The finite difference grid was the same. The timestep algorithm was the leap frog scheme. Every n time steps the fields were averaged according to

$$q\{(N+\tfrac{1}{2})\Delta t\} = 0.5\,[\,q\{N\Delta t\} + q\{(N+1)\Delta t\}\,]$$

$$q\{(N-\tfrac{1}{2})\Delta t\} = 0.5\,[\,q\{N\Delta t\} + q\{(N-1)\Delta t\}\,],$$

and the integration was restarted from the fields on the left. Sadourny used $\nu = 1/N$ as a measure of the dissipativity of the time differencing. On this measure the posential enstrophy scheme was much more stable than the energy scheme (Fig. 3).

Having found a critical ν_c for each model, he then integrated each model for a long time. In the energy conserving model Z, the potential enstrophy, grows very slowly for a long time after which there is a catastrophic change in the behaviour (Fig. 3). Z grows rapidly to a new equilibrium level and there is a marked energy dissipation.

Sadourny's comment was "the triad inter-actions in the smaller scales are far from accurate, their structure being essentially governed by the intrinsic conservation properties of the finite difference scheme. They produce three-dimension like energy exchanges in the absence of formal enstrophy conservation. The effect of the smaller scales becomes predominant around the critical time, accounting for the enstrophy increase".

The behaviour of the enstrophy conserving scheme is quite different (Fig. 4). Enstrophy dissipation is more marked than energy dissipation and both energy and enstrophy are very well conserved. At first sight this is a most surprising result.

However, the effect of the enstrophy conservation is to keep the energy in the longer wave lengths. These are in turn treated much more accurately so that the effects of non-conservation of energy are thereby reduced.

Fig. 5 and 6 show the equilibrium energy spectra for the rotational and divergent wind in the two models after a long period of integration. The spectra are quite different for the rotational wind and rather similar for the divergent wind. In the energy preserving model there is equipartition of the rotational energy. The rotational energy in the other model has a k^{-2} spectrum corresponding to an equipartition of enstrophy.

In both models the divergent energy is almost equi-partitioned.

There are reasons for believing that if a quantity is equi-partitioned in a truncated simulation such as Sadourny's, then in the real world the same quantity will be cascaded to smaller and smaller scales.

Thus, there are reasons for believing that in truncated two dimensional flow there may be a cascade of divergent energy into the dissipation range.

By the same arguments, the energy conserving model has a tendency to cascade energy to small scales while the enstrophy conserving has a tendency to cascade enstrophy to small scales.

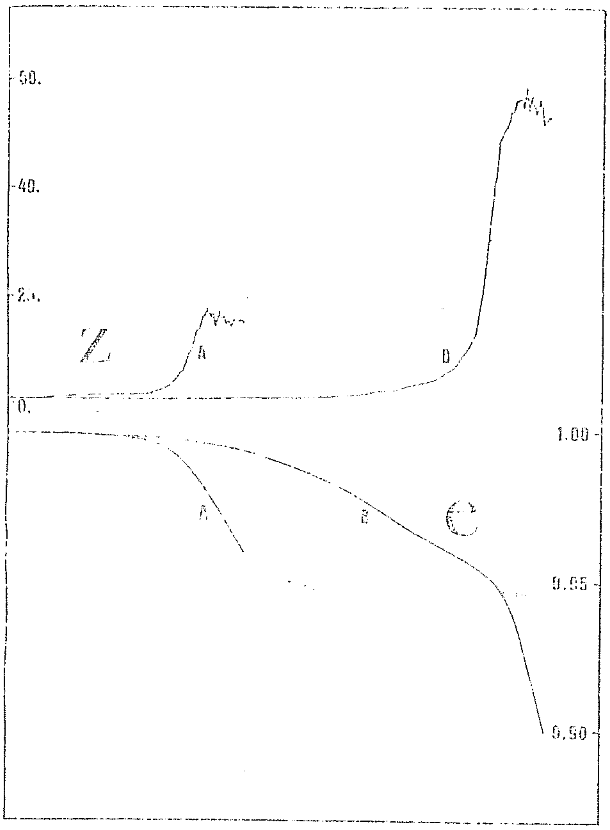

Fig. 3 Time evolution of potential enstrophy (Z) and energy (e) in the case of the energy-conserving model showing the influence of resolution: case A, 16×16 grid; case B, 32×32 grid (same initial conditions). Sadourny (1975)

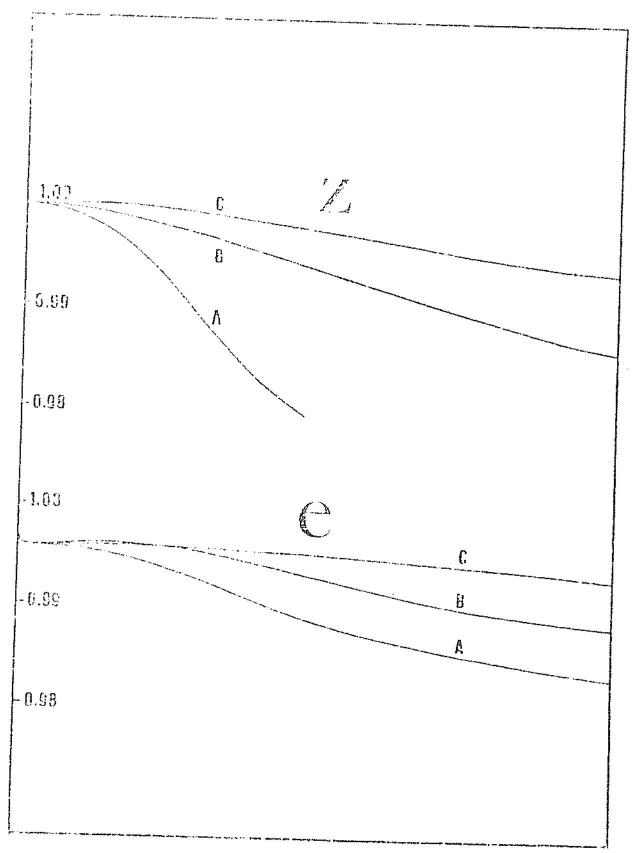

Fig. 4 Time evolution of potential enstrophy (Z) and energy (e) in the case of the potential enstrophy-conserving model: case A, $\nu=0.0025$; case B, $\nu=0.001$; case C, $\nu=0.00055$.
Sadourny (1975)

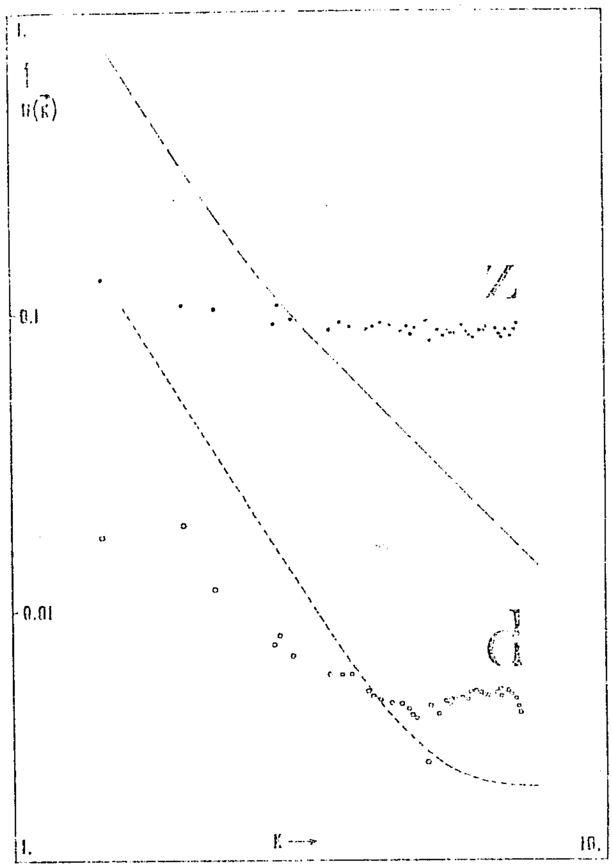

Fig. 5 Equilibrium energy spectra in the case of the quasi-inviscid energy-conserving model, showing average energy per vector mode as a function of the pseudo-wavenumber K. Z: rotational energy, d: divergent energy. Dashed lines indicate corresponding spectra in the case of the potential-enstrophy-conserving model, using the same value of dissipativity (ν = 0.005). Both coordinates are logarithmic. Sadourny (1975)

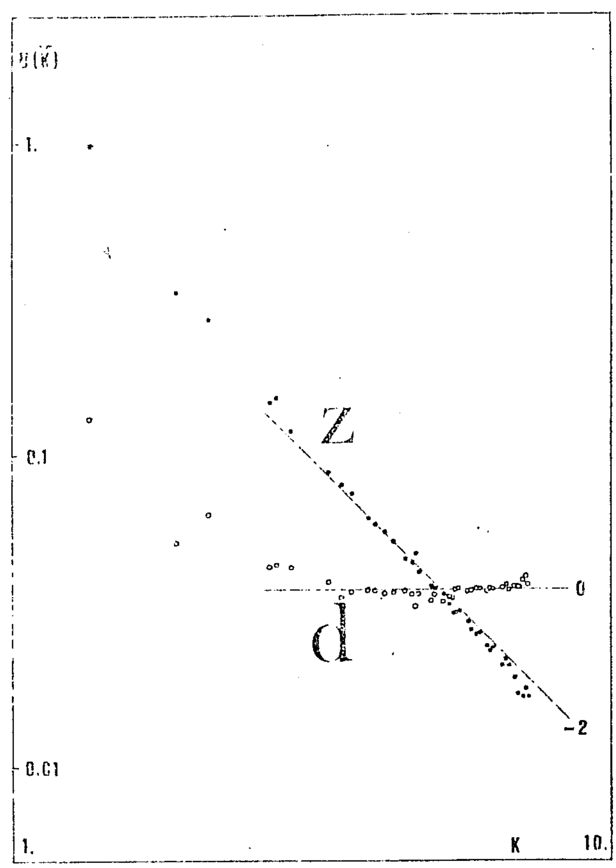

Fig. 6 Equilibrium energy spectra in the case of the quasi-inviscid potential-enstrophy-conserving model, showing equipartition of enstrophy and divergent energy in higher wavenumbers. Same conventions as in Fig. 7. Sadourny (1975)

This experiment indicates the profound importance of the correct formulation of conservation laws for long term integrations. It is still not clear if the conservation is equally crucial for medium range forecasts.

Eddy viscosity

As we have seen, the enstrophy conserving two dimensional adiabatic model will tend to cascade enstrophy into the shortest scales. If we are to represent the effect of the unresolved scales on the resolved scales, then we must represent the important features of the cascade by removing enstrophy at the shortest scales. Similarly, in three dimensional flow the tendency will be for a cascade of energy into the shortest scales, and this must be removed.

Kraichnan (1976) has studied the effects of a linear eddy viscosity using a turbulence closure model for isotropic turbulence known as the Test Field Model (TFM). With this model he could calculate analytically the equilibrium energy spectrum $E(k)$ and $T(k/k_m)$ the total rate of energy transfer per unit mass to all wave numbers above a given wave number k_m where the triad interactions involved (k, p, q) were such that one wave number, say, $k < k_m$ and p and/or $q > k_m$.

He defined an effective eddy viscosity acting on modes of wave number k due to dynamical interaction with wave numbers $> k_m$.

$$\nu(k|k_m) = -T(k|k_m)/[2k^2 E(k)] \quad k < k_m \qquad 12$$

If we are in the inertial range in 3 dimensions, then the eddy viscosity can depend only on the dissipation rate and the cut-off wave number, so that

$$\nu(k|k_m) = \beta_1 \varepsilon^{1/3} k_m^{-4/3} \qquad k \ll k_m \qquad 13$$

Fig.7 shows the normalised eddy viscosity

$$\nu(k|k_m)/\nu(0,k_m)$$

in the three dimensional inertial range. We see that the value is close to 1 for all waves such that $k < k_m/2$ i.e. for all waves longer than $4\Delta x$. Kraichnan concludes that if we are not concerned with accuracy in the wave-length range $2\Delta x$ to $4\Delta x$ then an eddy viscosity is apparently quite satisfactory. This wave-length range includes half of all possible wave-lengths.

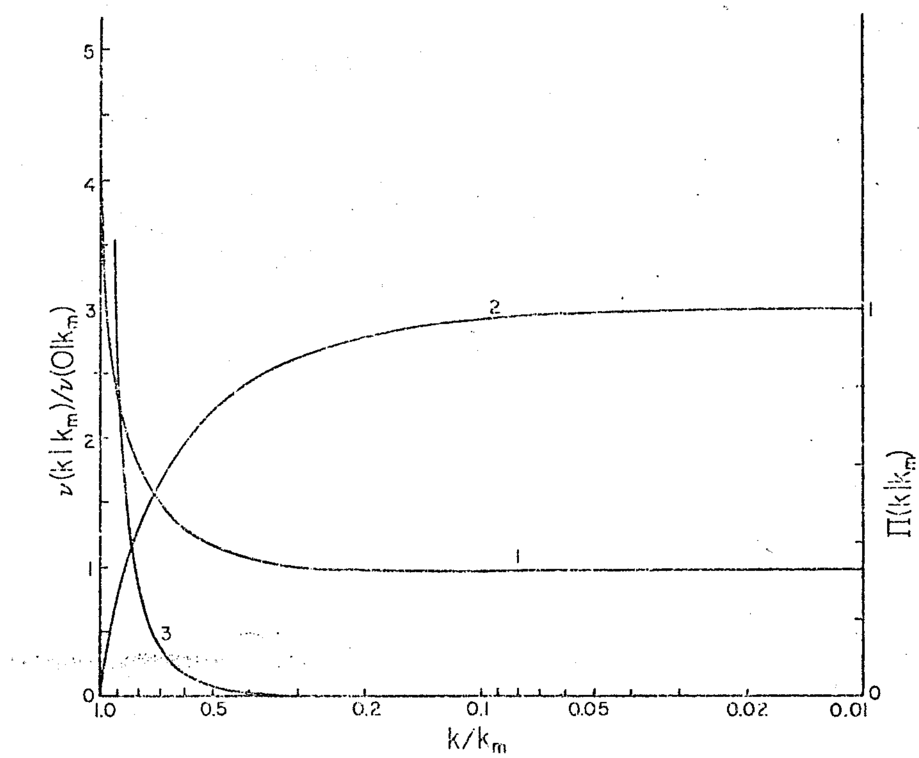

Fig. 7 Energy transfer across k_m in the three-dimensional inertial range. Curve 1, normalized eddy viscosity $\nu(k|k_m)/\nu(0|k_m)$; curve 2, integrated transfer $\Pi(k|k_m)$; curve 3, input contribution $-\nu_i(k|k_m)/\nu(0|k_m)$. Kraichnan (1976)

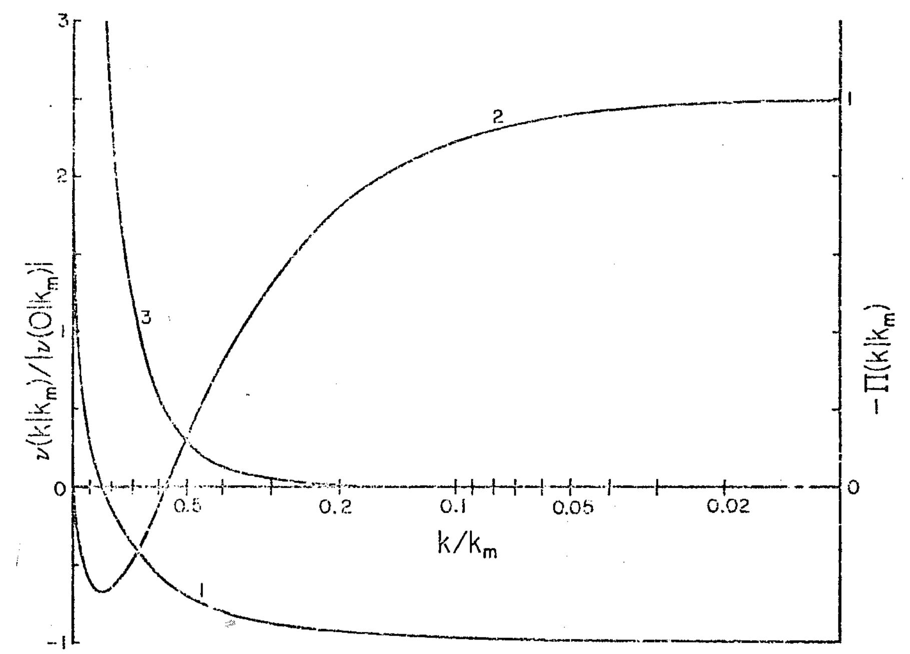

Fig. 8 Energy transfer across k_m in the two-dimensional energy inertial range. Curve 1, normalized eddy viscosity $\nu(k|k_m)/|\nu(0|k_m)|$; curve 2, integrated transfer $-\Pi(k|k_m)$; curve 3, input contribution $-\nu_i(k|k_m)/|\nu(0|k_m)|$. Kraichnan (1976)

Fig. 8 shows the effective eddy viscosity in two dimensional turbulence if k_m is in the energy inertial range. In this range, we expect the energy cascade to be towards longer waves and so it is no surprise to find that for almost all wave-lengths, the eddy viscosity is negative.

Finally, Kraichnan considers the suitability of an eddy viscosity formulation if the cut-off wave number is in the enstrophy cascading range of two dimensional turbulence. He finds that the general features are similar to those in Fig. 8 with some important differences.

1. if k_o is the (low) wave number characteristic of the wave lengths of excitation $(k_o << k_m)$ then the enstrophy transfer is concentrated in a region of width k_o at k_m,

2. the transfer is not proportional to the enstrophy intensity $\Omega(k)$ but to the derivative $d\Omega(k)/dk$

   The first feature means that the transfer curve does not scale with k_m alone but also with k_o, a parameter outside the inertial range. The second feature means that in x space the transfer cannot be well approximated by an operator in the form $\nu_{eddy} \nabla^2$. Instead the enstrophy transfer is better described as a diffusion process in wave number space.

Having considered some of the problems involved in representing the effect of unresolved scales in three dimensional, two dimensional, and quasi-geostrophic turbulence, we now take a look at what has actually been used in numerical forecasting and general circulation studies.

In early studies with numerical models the use of a ν eddy ∇^2 dissipation was found to have practical problems. The range of resolved scales was so short that a ν large enough to control the shortest wave-lengths had a severe damping effect on the longest resolved waves. In other words, the ∇^2 operator, which acts globally because of its linearity, is not sufficiently scale selective.

Smagorinsky (cf. Miyakoda 1973) presented a rather novel approach to the problem.

The Reynolds stress terms in the equations of motion may be written in the form:

$$\frac{\partial u_i}{\partial t} = \frac{\partial}{\partial x_j} \tau_{ij} + \ldots \qquad 14$$

$$\tau_{ij} = \overline{u_i' u_j'} \qquad 15$$

Since τ_{ij} is symmetric we may postulate

$$\tau_{ij} = K_{ij} S_{ij} \qquad 16$$

where S_{ij} is the deformation field, the symmetric part of the strain tensor.

If the turbulence is two dimensionally isotropic so that

$$K_{ij} = K_H \qquad \text{if } \begin{Bmatrix} i \neq 3 \\ j \neq 3 \end{Bmatrix}$$

$$K_{ij} = K_3 \qquad \text{otherwise}$$

then,

$$-\int \varepsilon \, dM = -\int K_H (D_T^2 + D_S^2) \, dM - \int K_3 \left[\left(\frac{\partial u_1}{\partial z}\right)^2 + \left(\frac{\partial u_2}{\partial z}\right)^2 \right] dM \qquad 17$$

where ε is the kinetic energy dissipation rate and

$$D_T = S_{11} = -S_{22} \, , \quad D_S = S_{21} = S_{12} \, ;$$

so that

$$\varepsilon_H \sim K_H (D_S^2 + D_T^2)$$

$$\varepsilon_V \sim K_3 (u_z^2 + v_z^2) \qquad 18$$

Now if we are in the energy cascade range of three dimensional turbulence then the only parameters that will determine the coefficient K_H are the energy dissipation rate and the cut-off wave number k_{H*}, k_{V*}.

Then, from dimensional considerations

$$K_H \propto \varepsilon_H^{1/3} (k_{H*})^{-4/3} , \quad K_V \propto \varepsilon_V^{1/3} (k_{V*})^{-4/3}$$

We now use the estimates (18) for the dissipation to eliminate the dissipation and find

$$K_H \propto (D_S^2 + D_T^2)^{1/2} \, k_{H*}^{-2}$$

$$K_V \propto (u_z^2 + v_z^2)^{\frac{1}{2}} \, k_{V*}^{-2} \qquad 19$$

or,

$$K_H = 2(k_0 \Delta s)^2 [D_S^2 + D_T^2]^{1/2} \qquad 20$$

$$K_V = (k_1 \Delta z)^2 [u_z^2 + v_z^2]^{1/2}$$

Leith (1972) considered the equivalent problem in two dimensional turbulence where the energy cascade is asymptotically zero and the dynamically relevant quantity is the enstrophy cascade rate η

Using a very similar argument to the above he found for two dimensional turbulence

that $K_H \propto (k_2 \Delta s)^3 |\nabla \zeta|$

where $\zeta = v_x - u_y$

Smagorinsky's formulation has been widely used but I have not been able to find any published reports on the use of the Leith formulation.

Corby, Gilchrist and Newson (1972) have used a formulation of the form

$$\frac{\partial \alpha}{\partial t} = --- + K_\alpha |\nabla^2 \alpha| \nabla^2 \alpha \qquad \alpha = u, v, T \qquad 21$$

The major difference between these formulations and the veddy ∇^2 formulation is the fact that the latter operators are non-linear and so tend to be local in their effects rather than global. In other words they reduce local extremes of curvature of deformation or vorticity gradient but have little effect where these quantities are small.

Williamson (1978) reports a series of experiments where he used an even more scale selective diffusion of the form $\nabla^2 K \nabla^2$ where K is the same as the K_H of Smagorinsky's formulation. He found that this had beneficial effects on long wave behaviour by confining the dissipation to the shortest waves.

To summarise then:

- The constraint of quasi-geostrophy imposes a conservation law on the three dimensional flow that is similar in form to the enstrophy conservation law for two-dimensional turbulence;

- provided that one can neglect the presence of boundaries, and the beta effect, that one can assume that non-linear interactions are local in wave-number space and that the turbulence is locally homogenous and isotropic, then one can argue the existence of an inertial range which cascades quasi-geostrophic potential enstrophy and such that the energy follows a k^{-3} power law;

- for three dimensional turbulence, Kraichnan's recent work indicates that a simple eddy viscosity may be quite adequate provided one does not need information in the $2\Delta x$ to $4\Delta x$ band of wave-lengths;

- for two dimensional turbulence when the cut-off wave length is in the enstrophy cascade range, Kraichnan's results indicate that an eddy viscosity formulation is likely to be quite unsatisfactory. Moreover, for wave-lengths much shorter than the cut-off wave-length, the eddy viscosity must be negative;

- the range of validity of the k^{-3} law for atmospheric flow is not clear. It seems unlikely that it can apply for scales less than 500km in mid latitudes. Direct measurement of the spectrum down to scales of 500 km may be possible during FGGE. For shorter scales, the law could perhaps be tested using a numerical model that conserved the Ertel potential vorticity;

- observations indicate that relatively little of the atmosphere's energy is dissipated internally, the bulk of the dissipation takes place at the surface boundary layer and in an internal boundary layer near the troposphere. The finite difference scheme presently used in the Centre's models should inhibit a cascade of energy into the smallest scales;

- in the absence of clear information on the expected structure of the atmospheric spectrum in the 100 km-500km range one should, with a 100km resolution, use a very scale-selective diffusion and one should make it as weak as possible and have the bulk of the dissipation done by the boundary layer and the region near the tropopause.

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