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ON INITIAL CONDITIONS FOR NONHYDROSTATIC MODELS

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Abstract

The problem of wave solutions in an isothermal, nonhydrostatic atmosphere at rest is considered. A linear perturbation analysis is performed. The solutions obtained under the assumption that the Coriolis parameter is constant consist of sound waves and gravity-inertial waves.

It is investigated how the hydrostatic assumption and an assumption of incompressibility influence the phase speed with the result that these assumptions are unjustified on a sufficiently small scale although both assumptions filter the sound waves.

The initial value problem of finding the partitioning of the initial amplitudes of the various parameters on the existing wave components is solved for the cases of a vanishing or a constant Coriolis parameter. It is possible to determine the initial conditions which will prevent any sound waves in a linear solution. The procedure is equivalent to the normal mode initialisation but definite relationships, depending on the eigenvalues, between the meteorological parameters are established.

1. Introduction

The purpose of this note is to investigate the impact of the hydrostatic and various other assumptions on the phase speed of harmonic waves. The model adopted for the study has a basic state characterised by isothermal conditions and no motion. The linearised equations include the effects of the pressure force, the Coriolis force and gravity. In treating the Coriolis force it has been assumed that the Coriolis parameter is constant (f - plane). The investigation is similar to one carried out by Herbert (1971) except that rotational effects have been included in this study in such a way that the validity of the results can be extended to larger scales.

The possible wave types in the general case are sound waves and gravity-inertia waves. The general frequency equation is therefore of the fourth degree. It can be solved exactly or by approximation, depending on the scale of the perturbation. Both methods are used. Since the general frequency equation is solvable it is straightforward to measure the impact of a given filter-approximation on the remaining waves. This is done separately for the hydrostatic assumption and an assumption of incompressibility. As expected, it turns out that these filter approximations are valid on a sufficiently large scale but failed to reproduce the true phase speed on the small scale.

Perturbations in an isothermal atmosphere is a classical problem which has been solved by Solberg (see Godske et al 1957). In this paper we shall concentrate on filter approximations.

2. The Model

We consider a state of rest under isothermal conditions with a temperature T. For the basic state we have hydrostatic equilibrium, i.e.

$$\frac{d\bar{p}}{dz} = - g\bar{\rho} \tag{2.1}$$

To simplify the perturbation equations we introduce the variable

$$\varepsilon = \frac{p'}{\bar{p}}, \qquad s = \frac{\rho'}{\bar{\rho}} \tag{2.2}$$

or

$$p = \bar{p} + p' = \bar{p}(1 + \varepsilon); \quad \rho = \bar{\rho} + \rho' = \bar{\rho}(1 + s) \tag{2.3}$$

With these notations, with $\kappa = Cp/Cv$ and assuming that the Coriolis parameter is constant we find the following perturbation equations

$$\frac{\partial u}{\partial t} = - RT\frac{\partial \varepsilon}{\partial x} + f_o v$$

$$\frac{\partial v}{\partial t} = - RT\frac{\partial \varepsilon}{\partial y} - f_o u$$

$$\frac{\partial W}{\partial t} = - RT\frac{\partial \varepsilon}{\partial z} + g(\varepsilon - s) \tag{2.4}$$

$$\frac{\partial \varepsilon}{\partial t} = - \kappa\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial W}{\partial z}\right) + \frac{g}{RT} W$$

$$\frac{\partial s}{\partial t} = - \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial W}{\partial z}\right) + \frac{g}{RT} W$$

We shall prescribe perturbations of the form $\exp\left[ikx + ily - i\tau t\right]$. Inserting in (2.4) it is possible to obtain a single equation in the amplitude W(z) for the vertical velocity. From each time derivative we get a term containing τ. For purposes of easy identification we shall mark some of these values. It is thus easy to identify the effect of neglecting a term.

We introduce, therefore, the notations $\tau(u)$, $\tau(v)$, $\tau(W)$, $\tau(\varepsilon)$ and $\tau(s)$. As an example we may say that if we set $\tau(W) = 0$ we have made the hydrostatic assumption. Similarly, if $\tau(u) = \tau(v) = 0$ we have made the geostrophic assumption for the perturbations. With these notations we get from (2.4) a differential equation for $W(z)$, the amplitude of the perturbation in the vertical velocity of the form

$$\frac{d^2W}{dz^2} - B\frac{dW}{dz} + EW = 0 \qquad (2.5)$$

in which

$$B = \frac{g}{RT} \; \frac{\tau(s) - \tau(\varepsilon) + \kappa\tau(s)}{\kappa\tau(s)} \qquad (2.6)$$

and

$$E = \frac{1}{\kappa RT}\left[\tau(w)\tau(\varepsilon) + \frac{g}{RT}\left(1 - \frac{\tau(\varepsilon)}{\tau(s)}\right) + \frac{k^2\tau(v) + \ell^2\tau(u))\;(\kappa RT\tau(w)\tau(s) + g^2(1-\kappa))}{\tau(s)\;(fo^2 - \tau(u)\tau(v))}\right] \qquad (2.7)$$

We note from (2.6) that in the case where $\tau(s)$ and $\tau(\varepsilon)$ are the same we find $B = g/(RT) = H_o^{-1}$ where H_o is the scale height for the isothermal atmosphere. Similarly, under the same conditions we see that the middle term in the bracket of (2.7) will vanish. The quantities B and E are constants under our conditions. It is thus straight forward to solve (2.5) under the boundary conditions that $W = o$ at $z = o$ and $z = H$. Let the solutions to the characteristic equations be $\lambda_1 \pm i\,\lambda_2$ where

$$\lambda_1 = \frac{1}{2}B \;;\; \lambda_2 = \frac{1}{2}(4E - B^2)^{\frac{1}{2}} \qquad (2.8)$$

giving

$$W = D_1 e^{\lambda_1 z} (\cos\lambda_2 z + i \sin\lambda_2 z) + D_2 e^{\lambda_1 z} (\cos\lambda_2 z - i \sin\lambda_2 z) \qquad (2.9)$$

$W = o$ at $z = o$ gives $D_1 + D_2 = o$ and

$$W = 2i\, D_1\, e^{\lambda_1 z} \sin \lambda_2 z \qquad (2.10)$$

whereafter $W = o$ at $z = H$ requires

$$\lambda_2 H = m\pi \qquad (2.11)$$

where $m = o, 1, 2, \ldots\ldots$ The frequency equation may therefore be written in the form

$$E = \frac{m^2\pi^2}{H^2} + \frac{1}{4} B^2 \qquad (2.12)$$

We consider first the general case in which all the five values of τ are the same. The notations will be simpler if we introduce

$$C_L^{\,2} = \kappa RT, \quad C_I^{\,2} = \frac{fo^2}{k^2 + l^2}, \quad N = 2\pi m \frac{Ho}{H} \qquad (2.13)$$

and

$$x = \frac{C}{C_L}, \qquad C = \frac{\tau}{\sqrt{k^2+l^2}} \qquad (2.14)$$

We find then

$$x^4 - \left[1 + \frac{C_I^{\,2}}{C_L^{\,2}} + \frac{1 + N^2}{4Ho^2(k^2+l^2)} \right] x^2 +$$

$$\left| \frac{C_I^{\,2}}{C_L^{\,2}} \frac{1 + N^2}{4H_o^{\,2}(k^2+\ell^2)} + \frac{\kappa - 1}{\kappa^2} \frac{1}{H_o^{\,2}(k^2+\ell^2)} \right| = 0 \qquad (2.15)$$

(2.15) is the most general frequency equation within the framework of this investigation. It will have four roots. Two of these will be numerically large and of opposite sign. They are the phase speeds of the sound waves in the model. The other two roots will be numerically smaller, but also of opposite sign. They are the phase speeds of the gravity-inertia waves in the model. We shall later discuss the numerical values of the solutions to the above equation.

The case of the hydrostatic assumption can be obtained from (2.5) - (2.7) by setting $\tau(W) = 0$ and $\tau(u) = \tau(v) = \tau(\varepsilon) = \tau(s)$. We find:

$$X_H^{\ 2} = \frac{C_I^{\ 2}}{C_L^{\ 2}} + 4\ \frac{\kappa - 1}{\kappa^2}\ \frac{1}{1 + N^2} \qquad (2.16)$$

The values of X_H obtained from (2.16) should be compared with the smaller roots obtained from (2.15).

For purposes of comparison it is also of interest to calculate the speed of pure sound waves in the model. This speed may be obtained by disregarding all reference to gravitational and rotational effects, i.e. $f = 0$ and $g = 0$. Under these conditions we find $B = 0$ and

$$E = \frac{1}{\kappa RT}\ (\tau^2 - (k^2 + \ell^2)\ \kappa RT) \qquad (2.17)$$

and from (2.12) that

$$X_S^{\ 2} = 1 + \frac{m^2 \Pi^2}{H^2 (k^2 + \ell^2)} \qquad (2.18)$$

(2.18) shows that X_S is always larger than unity except when $m = 0$. For small scale motion, i.e. when $(k^2 + \ell^2)$ is large, we find $X_S \simeq 1$.

We return now to (2.15). Since the equation is of 4th degree it is a little difficult to analyse the magnitude of the roots.

However, approximations to the roots can be obtained separately for perturbations on the very small or the very large scales. Considering first sufficiently small scales we note that

$$\frac{C_I^{\,2}}{C_L^{\,2}} + \frac{1 + N^2}{4H_o^{\,2}(k^2 + \ell^2)} << 1 \qquad (2.19)$$

because $k^2 + \ell^2$ will be large. Under these conditions we may write (2.15) in the form

$$X^4 - X^2 + \frac{\kappa-1}{\kappa^2}\frac{1}{H_o^{\,2}(k^2 + \ell^2)} = 0 \qquad (2.20)$$

From (2.19) it is seen that the inequality holds when the scale is small compared with 60km. This value is obtained for H_o = 7321m, H = 25000m, m=1, $C_L^{\,2} = 0.1 \times 10^6 m^2 s^{-2}$, $f = 10^{-4}\ s^{-1}$ and $k = \ell = 2\Pi_L^{\,-1}$. Under the same conditions it is seen that the first term in (2.19) is negligible compared to the second, saying that rotation is unimportant on this scale. The solution to (2.20) may be written in the form

$$X^2 \doteq \begin{cases} 1 - \dfrac{\kappa-1}{\kappa^2}\dfrac{1}{H_o^{\,2}(k^2 + \ell^2)} \\ \\ \dfrac{\kappa-1}{\kappa^2}\dfrac{1}{H_o^{\,2}(k^2 + \ell^2)} \end{cases} \qquad (2.21)$$

where we have used the first terms in the series expansion of the square root. The upper value is an approximation to the speed of the fast moving waves while the lower value applies to the slow moving gravity waves. We note that to this degree of approximation we find that the fast moving waves have a speed less than unity.

Considering next the large scale where

$$\frac{C_I^{\,2}}{C_L^{\,2}} + \frac{1 + N^2}{4H_o^{\,2}(k^2 + \ell^2)} >> 1 \tag{2.22}$$

we find that (2.15) reduces to

$$X^4 - \left(\frac{C_I^{\,2}}{C_L^{\,2}} + \frac{1 + N^2}{4H_o^{\,2}(k^2 + \ell^2)} \right) X^2 +$$

$$\left(\frac{C_I^{\,2}}{C_L^{\,2}} \frac{1 + N^2}{4H_o^{\,2}(k^2 + \ell^2)} + \frac{\kappa-1}{\kappa^2} \frac{1}{H_o^{\,2}(k^2 + \ell^2)} \right) = 0 \tag{2.23}$$

Using the same approximations as before it turns out that the roots to (2.23) can be written in the form

$$X_A^{\,2} = \begin{cases} \dfrac{1 + N^2}{4H_o^{\,2}(k^2 + \ell^2)} - 4 \dfrac{\kappa-1}{\kappa^2} \dfrac{1}{1 + N^2} \\[2ex] \dfrac{C_I^{\,2}}{C_L^{\,2}} + 4 \dfrac{\kappa-1}{\kappa^2} \dfrac{1}{1 + N^2} \end{cases} \tag{2.24}$$

The upper value in (2.24) is analogous to the result obtained by Herbert (1971). The same holds for the formulas given in (2.21). This agreement is to be expected because the results in (2.21) apply to very short waves where the Coriolis effect is unimportant and because the upper formula in (2.24) applies to sound waves. Since (2.24) applies to the large scale motion only, i.e. motions with a scale large compared to 60km, we may also state that the second term in the upper formula in (2.24) can be neglected because its value with the adopted parameters is about 0.2 while the first term exceeds this value for quite small values of the scale. Introducing this further approximation we may state that the speed of sound waves on the large scale in the model is given by

$$X_A = \frac{\sqrt{1 + N^2}}{2H_o} \cdot \frac{1}{\sqrt{k^2 + \ell^2}} \tag{2.25}$$

The values in (2.25) show good agreement on the large scale with the results obtained from (2.15) as can be seen from Table 1, which contain three different values of the speed of the sound waves. Table 1 shows in particular that X_S computed from (2.18) is an underestimate of the exact wave speed on the large scale. X_A from (2.25) is a good approximation to X_E down to scale of about 100 km.

We shall next compare the various estimates of the slowly moving gravity-inertial waves on the large scale. The speed of these waves has been computed from (2.15) (X_E), from the second expression in (2.24) (X_A) identical to (2.16) where the hydrostatic assumption was made in the equations. The comparisons can be made from Table 2 where we have included the values obtained if the Coriolis parameter is neglected (X_{HE}).

The value of the second term in (2.24) is, as mentioned, about 0.2. The first term in (2.24) can be neglected compared with the second if the scale is small, but still large enough that (2.24) is valid. Both terms have, however, been included in calculating X_A.

Table 2 shows that X_A is a good approximation to X_E down to a scale of almost 100 km (L=0.1). The same is true for the values obtained when the hydrostatic assumption is made from the outset because (2.16) is identical to the second expression in (2.24). If rotation is disregarded in the model as was done by Herbert (1971), we find that X_{HE} is close to X_E for scales around 100 km, but the deviations are large for larger scales. The level of a 10% difference between X_E and X_{HE} is reached at a scale of about 6000 km (L=6).

For the very short waves for which (2.21) applies we find that the approximation is less accurate. This is first of all demonstrated by Table 3 giving the exact and approximate values of the speed of the sound waves. We note first of all that the approximate values obtained from (2.21) are all less than unity while the exact values are larger than unity. The approximation is accurate only on the very small scales, i.e. less than about 30km when the error is 10%.

Table 4 applies to the gravity waves. As with the sound waves it is seen that X_A is a good approximation to X_E only on very small scales. The speeds based on the hydrostatic assumption are also a poor approximation to X_E on this scale.

All the tables have been prepared with $k = \ell$. To supplement the tables we have prepared figures showing the percentage difference between the exact speed and the speed based on the hydrostatic assumption, i.e.

$$Q = \frac{|X_E - X_H|}{X_E} \times 100 \qquad (2.26)$$

Figure 1 shows Q as a function of L_x for $L_y = 0.01$ and $m = 1$. For large values of L_x we find that Q is about 350%. Figure 2 shows Q as a function of L_x for $L_y = 0.1$ and $L_y = 1.0$ with $m = 1$. In these cases we find that Q approaches 8% for large L_x when $L_y = 0.1$ while Q for practical purposes vanishes for large L_x when $L_y = 1.0$. From these figures we may conclude that the hydrostatic assumption should not be applied when either L_x or L_y are small, say less than 100 km.

3. Incompressible Flow

It is well known that the incompressibility assumption will filter the sound waves. We shall make this assumption and investigate the speed of the remaining slow waves in comparison with the exact solution for the gravity-inertial waves. The basic system of equations are:

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= - RT \frac{\partial \varepsilon}{\partial x} + fv \\
\frac{\partial v}{\partial t} &= - RT \frac{\partial \varepsilon}{\partial y} - fu \\
\frac{\partial w}{\partial t} &= - RT \frac{\partial \varepsilon}{\partial z} + g (\varepsilon - s) \\
\frac{\partial s}{\partial t} &= \frac{g}{RT} W \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= o
\end{aligned}
\tag{3.1}
$$

in which the notations are the same as in (2.4). Employing perturbations of the same type as in section 2 we may derive an equation equivalent to (2.5). We get:

$$\frac{d^2W}{dz^2} - F \frac{dW}{dz} + G W = o \tag{3.2}$$

where

$$F = \frac{g}{RT} = \frac{1}{Ho} \quad ; \quad G = \frac{(k^2\tau(v) + \ell^2\tau(u))(\kappa RT\tau(w)\tau(p) - \kappa g^2)}{(f_o{}^2 - \tau(u)\tau(v)) \kappa RT\tau(\rho)} \tag{3.3}$$

In analogy with (2.12) the general frequency equation is

$$G = \frac{m^2\Pi^2}{H^2} + \frac{1}{4} F^2 \tag{3.4}$$

We consider first the general case in which $\tau(u)=\tau(v)=\tau(\rho)=\tau(W)$. Using the notations defined in (2.13) and (2.14) we find, after some rearrangements:

$$X^2 = \frac{1}{1 + \frac{4H_o^2\,(k^2 + \ell^2)}{1 + N^2}} \left(\frac{C_I^{\,2}}{C_L^{\,2}} + \frac{4}{\kappa}\,\frac{1}{1 + N^2} \right) \tag{3.5}$$

It is clearly seen from (3.5) that the sound waves have been filtered out by the assumption of incompressibility. If we furthermore make the hydrostatic assumption which can be done by setting $\tau(W)=0$ and $\tau(u)=\tau(v)=\tau(\rho)$ we find from (3.4) that

$$X_H^{\;2} = \frac{C_I^{\,2}}{C_L^{\,2}} + \frac{4}{\kappa}\,\frac{1}{1 + N^2} \tag{3.6}$$

Comparing (3.5) and (3.6) it is seen that the hydrostatic assumption in an incompressible atmosphere is permissible if the quantity

$$r = \frac{4H_o^{\,2}(k^2 + \ell^2)}{1 + N^2} \tag{3.7}$$

is close to zero, or if $(1 + r)^{-1}$ is close to unity. Figure 3 shows $(1 + r)^{-1}$ as a function of scale for $k = \ell$. It is seen that $(1 + r)^{-1}$ is larger than 0.9 when the scale is larger than about 200km. Practically no difference will exist between the values of X computed from (3.5) and (3.6) if the scale is larger than 1000km.

A comparison of (2.16) and (3.6) shows that the assumption of incompressible flow introduces a systematic difference between the speeds of the slowly moving waves under hydrostatic conditions. From the two equations we find that

$$X_{HI}^{\;2} - X_{HC}^{\;2} = \frac{4}{\kappa^2}\,\frac{1}{1 + N^2} = 0.46, \quad m = 1 \tag{3.8}$$

showing that the speed in the incompressible model is systematically larger than in the compressible model.

4. Initialisation for Non-Hydrostatic Motion

In this section we shall investigate the problem of integrating the non-hydrostatic equations with the assumption that it is desirable to eliminate the sound waves from the integration. As seen in the previous sections the largest differences between the phase speeds in non-hydrostatic and hydrostatic models occur on the smallest scales. At these scales we find furthermore that the Coriolis effect is of minor importance. It is therefore not unreasonable to start with a model where rotation is disregarded. In the next section we shall look at the effect of rotation. The problem without rotation has been considered by Herbert (1971) but his goal was to find a set of filtered equations which would describe the motion of the gravity waves with good accuracy and at the same time filter the sound waves from the system. We shall consider the problem from the point of view that we shall integrate the non-hydrostatic equations but attempt to initialise the state of the model in such a way that the sound waves are absent from the initial state. This is possible in the linear problem and if the linear equations were integrated the sound waves would not appear. Due to non-linear interactions the sound waves would be created in an integration of non-linear equations because the initialisation procedure is based on the linear equations.

Using the same notations as before and introducing $\mu = \varepsilon - s$ we find, disregarding the Coriolis effect and the meridional velocity, the following linear equations:

$$\frac{\partial u}{\partial t} = -\ RT\frac{\partial \varepsilon}{\partial x}$$

$$\frac{\partial w}{\partial t} = -\ RT\frac{\partial \varepsilon}{\partial z} + g\mu \qquad (4.1)$$

$$\frac{\partial \varepsilon}{\partial t} = -\ \kappa\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right) + \frac{1}{H_0}W$$

$$\frac{\partial \mu}{\partial t} = -\ (\kappa - 1)\left(\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z}\right)$$

Introducing perturbations of the form $\exp\left[ik(x - ct)\right]$ and eliminating all variables but $\hat{W} = \hat{W}(z)$ we find the following equation

$$\frac{d^2\hat{W}}{dz^2} - \frac{1}{H_0}\frac{d\hat{W}}{dz} + \left[\frac{g^2(\kappa - 1)}{C_L^4\, x^2} + k^2(x^2 - 1)\right]\hat{W} = 0 \qquad (4.2)$$

$$x = \frac{C}{C_L}, \qquad C_L^{\,2} = \kappa RT \qquad (4.3)$$

(4.2) is naturally equivalent to the equation derived by Herbert (1971), and we may adopt his solutions with the same boundary conditions, i.e. $\hat{W} = 0$ at $z = 0$ and $Z = H$. We find

$$X^2 = \frac{1}{2}\left(1 + \frac{1 + N^2}{4k^2H_0^{\,2}}\right) \pm \frac{1}{2}\left[\left(1 + \frac{1 + N^2}{4k^2H_0^{\,2}}\right)^2 - 4\frac{(\kappa - 1)}{\kappa^2}\,\frac{1}{H_0^{\,2}k^2}\right]^{\frac{1}{2}} \qquad (4.4)$$

The two roots corresponding to the plus sign in (4.4) are the sound wave phase speeds while the remaining solutions corresponding to the minus sign are the internal gravity waves. We shall denote the first roots by X_1 and $-X_1$, while the second set will be denoted X_3 and $-X_3$. For short waves, i.e. for

$$\frac{1 + N^2}{4k^2 H_0^2} << 1 \qquad (4.5)$$

we find

$$X_1^2 \simeq 1 - \frac{\kappa - 1}{\kappa^2} \frac{1}{H_0^2 k^2}$$

$$X_3^2 \simeq \frac{\kappa - 1}{\kappa^2} \frac{1}{H_0^2 k^2} \qquad (4.6)$$

while the approximation for long waves, i.e. for

$$\frac{1 + N^2}{4k^2 H_0^2} >> 1 \qquad (4.7)$$

is

$$X_1^2 \simeq \frac{1 + N^2}{4k^2 H_0^2} - 4 \frac{\kappa - 1}{\kappa^2} \frac{1}{1 + N^2}$$

$$X_3^2 \simeq 4 \frac{\kappa - 1}{\kappa^2} \frac{1}{1 + N^2} \qquad (4.8)$$

Returning to the basic equations we note that the solution for W can be written in the form

$$W = \sum_{n=1}^{4} W_n \, e^{\frac{z}{2H_0}} \sin\left(\frac{m\pi}{H} z\right) \cos\left(k(x - c_n t)\right) \qquad (4.9)$$

where n denotes the four solutions to the equations.

The next goal is to express the variables u, ε and μ in terms of W. From (4.1) using the form of the perturbations and a process of elimination we find that

$$\hat{u} = -i \frac{RT}{k} \frac{1}{C_L^2 - C^2} \left[\frac{1}{H_0} \hat{W} - \kappa \frac{d\hat{W}}{dz} \right]$$

$$\hat{\varepsilon} = -i \frac{c}{k} \frac{1}{C_L^2 - C^2} \left[\frac{1}{H_0} \hat{W} - \kappa \frac{d\hat{W}}{dz} \right] \qquad (4.10)$$

$$\hat{\mu} = -i \frac{\kappa - 1}{kc} \frac{1}{C_L^2 - C^2} \left[g \hat{W} - C^2 \frac{d\hat{W}}{dz} \right]$$

Using (4.9) it is then straightforward to calculate the form of the three variables. After some calculations we find that

$$u = \sum_{n=1}^{4} \frac{RT}{k} \frac{1}{C_L^2 - Cn^2} \frac{Wn}{2H_0} S(z) \sin (k(x - c_n t))$$

$$(4.11)$$

$$\varepsilon = \sum_{n=1}^{4} \frac{Cn}{k} \frac{1}{C_L^2 - Cn^2} \frac{Wn}{2H_0} S(z) \sin (k(x - c_n t))$$

where

$$S(z) = (2 - \kappa) \sin \left(\frac{m\pi}{N} Z\right) - \kappa N \cos \left(\frac{m\pi}{N} Z\right) \qquad (4.12)$$

For the last parameter μ we find

$$\mu = \sum_{n=1}^{4} \left[\frac{\kappa - 1}{kCn} \frac{2RT - Cn^2}{C_L^2 - Cn^2} \frac{Wn}{2H_0} e^{\frac{Z}{2H_0}} \sin\left(\frac{m\pi}{H} Z\right) \right. \tag{4.13}$$

$$\left. - \frac{\kappa - 1}{k} N \frac{Cn}{C_L^2 - Cn^2} \frac{Wn}{2H_0} e^{\frac{Z}{2H_0}} \cos\left(\frac{m\pi}{H} Z\right) \right] \sin(k(x - c_n t))$$

(4.11) and (4.13) give the functional form in which the parameter will occur in the solution. By setting $t = 0$ we can also see the way in which the initial conditions have to be given. It is important to note at this point that the co-efficient in the last term in (4.13) can be obtained from the co-efficient in ε by multiplication by the factor $-(\kappa - 1)N$. It follows, therefore, that the initial conditions must be given in the form:

$$w = W_0 \, e^{\frac{Z}{2H_0}} \sin\left(\frac{m\pi}{H} Z\right) \cos(kx)$$

$$\varepsilon = E_0 \, S(z) \sin(kx)$$

$$u = U_0 \, S(z) \sin(kx) \tag{4.14}$$

$$\mu = e^{\frac{Z}{2H_0}} \left[M_0 \sin\left(\frac{m\pi}{H} Z\right) - (\kappa - 1) N \, E_0 \cos\left(\frac{m\pi}{H} Z\right) \right] \sin(kx)$$

We observe next that (4.9), (4.11) and (4.13) for $t = 0$ must agree with the expressions (4.14). It is thus seen that we can obtain a set of four inhomogeneous linear equations for the determinations of $Wn, n = 1,2,3,4$. To write these equations in a convenient form we introduce the notations -

$$F_1(u) = \frac{1}{2\kappa H_0 k} \frac{1}{1 - x_1^2} \; ; \; F_2(u) = F_1(u)$$

$$F_3(u) = \frac{1}{2\kappa H_0 k} \frac{1}{1 - x_3^2} \; ; \; F_4(u) = F_3(u)$$

$$F_1(\varepsilon) = \frac{1}{2H_0 kC_L} \frac{x_1}{1 - x_1^2} \; ; \; F_2(\varepsilon) = - F_1(\varepsilon)$$

$$F_3(\varepsilon) = \frac{1}{2H_0 kC_L} \frac{x_3}{1 - x_3^2} \; ; \; F_4(\varepsilon) = - F_3(\varepsilon) \tag{4.15}$$

$$F_1(\mu) = \frac{\kappa - 1}{2\kappa H_0 kC_L} \frac{2 - \kappa x_1^2}{1 - x_1^2} \frac{1}{x_1} \; ; \; F_2(\mu) = -F_1(\mu)$$

$$F_3(\mu) = \frac{\kappa - 1}{2\kappa H_0 kC_L} \frac{2 - \kappa x_3^2}{1 - x_3^2} \frac{1}{x_3} \; ; \; F_4(\mu) = -F_3(\mu)$$

With the notations (4.15) we can write the four equations in the form:

$$\begin{array}{lllll} & W_1 + & W_2 + & W_3 + & W_4 = W_0 \\ F_1(u) & W_1 + F_1(u) & W_2 + F_3(u) & W_3 + F_3(u) & W_4 = U_0 \\ F_1(\varepsilon) & W_1 - F_1(\varepsilon) & W_2 + F_3(\varepsilon) & W_3 - F_3(\varepsilon) & W_4 = E_0 \\ F_1(\mu) & W_1 - F_1(\mu) & W_2 + F_3(\mu) & W_3 - F_3(\mu) & W_4 = M_0 \end{array} \tag{4.16}$$

It is straightforward to solve the system (4.16) by solving the first two equations for $(W_1 + W_2)$ and $(W_3 + W_4)$ and the last two equations for $(W_1 - W_2)$ and $(W_3 - W_4)$. The final result is with

$$\Delta_1 = 2(F_3(u) - F_1(u))$$

$$\Delta_2 = 2\left[F_1(\varepsilon)F_3(\mu) - F_1(\mu)F_3(\varepsilon)\right] \tag{4.17}$$

$$W_1 = \frac{F_3(u)W_0 - U_0}{\Delta_1} + \frac{F_3(\mu)E_0 - F_3(\varepsilon)M_0}{\Delta_2}$$

$$W_2 = \frac{F_3(u)W_0 - U_0}{\Delta_1} - \frac{F_3(\mu)E_0 - F_3(\varepsilon)M_0}{\Delta_2} \qquad (4.18)$$

$$W_3 = \frac{U_0 - F_1(u)W_0}{\Delta_1} - \frac{F_1(\mu)E_0 - F_1(\varepsilon)M_0}{\Delta_2}$$

$$W_4 = \frac{U_0 - F_1(u)W_0}{\Delta_1} + \frac{F_1(\mu)E_0 - F_1(\varepsilon)M_0}{\Delta_2}$$

(4.18) is the general solution to the initial value problem. For given initial conditions (W_0, U_0, E_0 and M_0) we may calculate the amplitudes W_n, $n = 1,2,3,4$ and from those the corresponding amplitudes of the four components in U_n, E_n and M_n. The rather cumbersome calculations leading to (4.18) have been carried out because it is possible to state exactly how the initial conditions have to be specified to avoid any sound waves in the linear model. (4.18) may furthermore be used to investigate how large the amplitude of the sound waves will be if we deviate from these exact conditions. Since the indices 1 and 2 indicate the sound waves it is obviously desirable to adjust the initial conditions in such a way that $W_1 = W_2 = 0$. It is seen that these conditions are satisfied if

$$U_0 = F_3(u)W_0; \; E_0 = \frac{F_3(\varepsilon)}{F_3(\mu)} M_0 \qquad (4.19)$$

Rewriting the first relation in (4.19) using the expression for $F_3(u)$ we find that

$$W_0 = 2\kappa H_0 k(1 - X_3{}^2)U_0 \qquad (4.20)$$

For short waves we may write (4.20) in the form

$$W_0 = 2\kappa H_0 k\left(1 - \frac{\kappa - 1}{\kappa}\,\frac{1}{H_0{}^2 k^2}\right)U_0 \qquad (4.21)$$

using the expression for $X_3{}^2$ from (4.6). For long waves we apply (4.8) and obtain

$$W_0 = 2\kappa H_0 k\left(1 - 4\,\frac{\kappa - 1}{\kappa^2}\,\frac{1}{1 + N^2}\right)U_0 \qquad (4.22)$$

A comparison between (4.20) and (4.21) shows that (4.21) gives good accuracy up to about 40km. Similarly, a comparison between (4.22) and (4.20) shows that the long wave formula is a good approximation about 50km.
From the second part of (4.19) we obtain

$$E_0 = \frac{\kappa}{\kappa - 1}\,\frac{X_3{}^2}{2 - \kappa X_3{}^2}\,M_0 \qquad (4.23)$$

which for short waves is approximated by

$$E_0 = \frac{1}{2\kappa H_0{}^2 k^2 - (\kappa - 1)}\,M_0 \qquad (4.24)$$

while the approximation for long waves is

$$E_0 = \frac{2}{\kappa(1 + N^2) - 2(\kappa - 1)}\,M_0 \qquad (4.25)$$

The main result of the investigation so far is that (4.19) or, equivalently, (4.20) and (4.23) give the exact conditions which will guarantee that the amplitudes of the sound waves vanish in the linear model.

We shall next investigate what would happen if we imposed the hydrostatic assumption on the initial data. As seen from the second equation in (4.1) we would then have the relation

$$u = H_0 \frac{\partial \varepsilon}{\partial z} \qquad (4.26)$$

for the initial data leading to a relation between M_0 and E_0. We may find this relation by inserting the solution (4.14) for t = 0 in (4.26). After some calculations we find that the resulting relation is identical to (4.25) which shows that the hydrostatic assumption can be used as an initial constraint with good accuracy on the large scale.

It is naturally well known that the hydrostatic assumption implies Richardson's equation which is obtained by requiring that $\partial^2 W/\partial t^2 = 0$ which in turn implies that the equation obtained by time differentiation of (4.26) is valid. Substituting in this equation from (4.1) we find that

$$\frac{\partial^2 W}{\partial Z^2} - \frac{1}{H_0}\frac{\partial W}{\partial Z} = (\kappa - 1)\frac{1}{\kappa H_0}\frac{\partial u}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) \qquad (4.27)$$

or

$$\frac{d^2\hat{W}}{dZ^2} - \frac{1}{H_0}\frac{d\hat{W}}{dz} = \frac{\kappa - 1}{\kappa}\frac{k}{H_0}\hat{u} - k\frac{d\hat{u}}{dz} \qquad (4.28)$$

Since (4.28) must be satisfied by the solutions given in (4.14) it is seen that by substitution from (4.14) in (4.28) we will obtain a relation between W_0 and U_0.

After some calculations it turns out that the resulting relation from (4.28) is identical to (4.22) which was the approximation to the exact calculation for long waves. We arrive thus at the same conclusions as earlier that the hydrostatic relation imposed on the initial state will approximate the exact relations for vanishing sound waves on the large scale. The hydrostatic relation will, however, not eliminate the sound waves from the numerical integration.

To illustrate the validity of the hydrostatic assumption the ratios $E_0/E_{0,H}$ and $W_0/W_{0,H}$ have been computed as a function of wavelength. The results are given in Figure 4 showing that these ratios are close to unity for long waves but deviate considerably from unity for short waves.

5. The Initial Value Problem

We shall return to the model treated in sections 2 and 3. The major problem to be investigated will be the initial conditions which must be imposed either to eliminate the sound waves completely or to reduce the amplitudes of these waves to such an extent that they are harmless. The technique will be to fit the solutions obtained earlier to arbitrary initial conditions and then to select these in such a way that the sound waves are (almost) eliminated. It turns out to be most convenient to replace the two equations of motion with the ordinary vorticity and divergence equation. If we, in addition, introduce the parameter $\mu=\varepsilon - s$ we get the equations

$$\frac{\partial \zeta}{\partial t} = - f_o D$$

$$\frac{\partial D}{\partial t} = - RT\nabla^2 \varepsilon + f_o \zeta$$

$$\frac{\partial W}{\partial t} = - RT \frac{\partial \varepsilon}{\partial z} + g\mu \tag{5.1}$$

$$\frac{\partial \varepsilon}{\partial t} = - \kappa\left(D + \frac{\partial W}{\partial z}\right) + \frac{1}{H_o} W$$

$$\frac{\partial \mu}{\partial t} = - (\kappa - 1)\left(D + \frac{\partial W}{\partial z}\right)$$

These equations are equivalent to the system (2.4). They lead to the frequency equation (2.5) and to the solution (2.15). The form of the solution for the vertical velocity may, in agreement with (2.10), be written in the form

$$W = \sum_{n=1}^{4} W_n \, e^{\frac{z}{2Ho}} \sin\left(\frac{m\Pi}{H} z\right) \cos(kx + \ell y - \tau t) \tag{5.2}$$

With the solution (5.2) we may use (5.1) to derive the form of the solution for the other parameters recalling that all solutions are of the form: $G(z)\exp[i(kx + \ell y - \tau t)]$ when $G(z)$ has to be determined for each parameter and furthermore that the parameters have to be real. After straightforward eliminations we find that

$$\varepsilon = \sum_{n=1}^{4} E_n \Psi(z) \sin(kx + \ell y - \tau t)$$

$$D = \sum_{n=1}^{4} D_n \Psi(z) \cos(kx + \ell y - \tau t) \tag{5.3}$$

$$\zeta = \sum_{n=1}^{4} Z_n \Psi(z) \sin(kx + \ell y - \tau t)$$

where

$$\Psi(z) = \frac{1}{2H_o} e^{\frac{z}{2H_o}} \left\{ (2-\kappa)\sin\left(\frac{m\Pi}{H}z\right) - \kappa N \cos\left(\frac{m\Pi}{H}z\right) \right\} \tag{5.4}$$

and where

$$E_n = -\frac{1}{\tau_n} \frac{f_o^2 - \tau_n^2}{f_o^2 + \kappa RT(k^2+\ell^2) - \tau_n^2} W_n = -F_n(\varepsilon) W_n$$

$$D_n = \frac{RT(k^2+\ell^2)}{f_o^2 + \kappa RT(k^2+\ell^2) - \tau_n^2} W_n = F_n(D) W_n \tag{5.5}$$

$$Z_n = \frac{f_o}{\tau_n} \frac{RT(k^2+\ell^2)}{f_o^2 + \kappa RT(k^2+\ell^2) - \tau_n^2} W_n = F_n(\zeta) W_n$$

For the last parameter μ we find that it has the form

$$\mu = \sum_{n=1}^{4} \left[\frac{\kappa-1}{f_o H_o} Z_n \sin\left(\frac{m\Pi}{H}z\right) - (\kappa-1) E_n \frac{1}{2H_o} \left\{ \sin\left(\frac{m\Pi}{H}z\right) + N\cos\left(\frac{m\Pi}{H}z\right) \right\} \right] e^{\frac{z}{2H_o}} \sin(kx + \ell y - \tau t) \tag{5.6}$$

It is thus seen that due to (5.6) we are not permitted to prescribe μ with an arbitrary co-efficient.
The initial conditions must naturally be prescribed in such a way that their form is in agreement with the above solutions. We shall therefore write at t = o.

$$
\begin{aligned}
w &= W_o\, e^{\frac{z}{2H_o}} \sin(\frac{m\Pi}{H} z) \cos\,(kx + \ell y) \\
\varepsilon &= E_o\, \Psi\,(z)\, \sin\,(kx + \ell y) \\
D &= D_o\, \Psi\,(z)\, \cos\,(kx + \ell y) \\
\zeta &= Z_o\, \Psi\,(z)\, \sin\,(kx + \ell y)
\end{aligned}
\qquad (5.7)
$$

The expressions in (5.7) shall be equal to the corresponding expressions in (5.2) and (5.3) setting t = 0. This procedure results in four equations in the four unknowns W_n, n=1,2,3,4. To write these equations in a simple form we note that

$$
\begin{aligned}
&F_1(\varepsilon) = -F_2(\varepsilon),\ F_3(\varepsilon) = -\ F_4(\varepsilon) \\
&F_1(D) = F_2(D),\ F_3(D) = F_4(D) \\
&F_1(\zeta) = -F_2(\zeta),\ F_3(\zeta) = -F_4(\zeta)
\end{aligned}
\qquad (5.8)
$$

because $\tau_1 = -\tau_2$ and $\tau_3 = -\tau_4$. In writing these results we have used τ_1 and τ_2 to denote the frequencies of the sound waves while τ_3 and τ_4 are those for the gravity-inertial waves. With these notations we have

$$
\begin{aligned}
W_1 + W_2 + W_3 + W_4 &= W_o \\
-F_1(\varepsilon)\, W_1 + F_1(\varepsilon)\, W_2 - F_3(\varepsilon)\, W_3 + F_3(\varepsilon)\, W_4 &= E_o \\
F_1(D)\, W_1 + F_1(D)\, W_2 + F_3(D)\, W_3 + F_3(D)\, W_4 &= D_o \\
F_1(\zeta)\, W_1 - F_1(\zeta)\, W_2 + F_3(\zeta)\, W_3 - F_3(\zeta)\, W_4 &= Z_o
\end{aligned}
\qquad (5.9)
$$

The solution to (5.9) is:

$$
\begin{aligned}
W_1 &= \frac{1}{2}\left[\frac{F_3(D)W_0 - D_0}{F_3(D) - F_1(D)} + \frac{F_3(\zeta)E_0 + F_3(\varepsilon)Z_0}{F_1(\zeta)F_3(\varepsilon) - F_1(\varepsilon)F_3(\zeta)}\right] \\
W_2 &= \frac{1}{2}\left[\frac{F_3(D)W_0 - D_0}{F_3(D)-F_1(D)} - \frac{F_3(\zeta)E_0 + F_3(\varepsilon)Z_0}{F_1(\zeta)F_3(\varepsilon) - F_1(\varepsilon)F_3(\zeta)}\right] \\
W_3 &= \frac{1}{2}\left[\frac{D_0 - F_1(D)W_0}{F_3(D) - F_1(D)} - \frac{F_1(\zeta)E_0 + F_1(\varepsilon)Z_0}{F_1(\zeta)F_3(\varepsilon) - F_1(\varepsilon)F_3(\zeta)}\right] \\
W_4 &= \frac{1}{2}\left[\frac{D_0 - F_1(D)W_0}{F_3(D) - F_1(D)} \quad \frac{F_1(\zeta)E_0 + F_1(\varepsilon)Z_0}{F_1(\zeta)F_3(\varepsilon) - F_1(\varepsilon)F_3(\zeta)}\right]
\end{aligned}
\tag{5.10}
$$

Recalling that W_1 and W_2 are the amplitudes related to the sound waves and that for meteorological calculations it is most often wanted to reduce them to zero we note that this is accomplished if

$$
\begin{aligned}
D_0 &= F_3(D)\, W_0 \\
E_0 &= -\frac{F_3(\varepsilon)}{F_3(\zeta)}\, Z_0
\end{aligned}
\tag{5.11}
$$

We notice that the first equation of (5.11) may be written in the form:

$$W_0 = \kappa\left(1 + \frac{C_I^{\,2}}{C_L^{\,2}} - X_3^{\,2}\right) D_0 \tag{5.12}$$

X_3 is determined from (2.15). From (2.21) we find for the small scale that

$$W_0 = \kappa\left(1 + \frac{C_I^{\,2}}{C_L^{\,2}} - \frac{\kappa - 1}{\kappa^2}\,\frac{1}{H_0^{\,2}(k^2 + \ell^2)}\right) D_0 \tag{5.13}$$

or, with good approximation,

$$W_0 = \kappa\left(1 - \frac{\kappa - 1}{\kappa^2}\,\frac{1}{H_0^{\,2}(k^2 + \ell^2)}\right) D_0 \tag{5.14}$$

On the other hand, for the large scale we have (2.24) giving

$$W_0 = \kappa(1 - \frac{\kappa - 1}{\kappa^2} \frac{4}{1 + N^2} D_0 \qquad (5.15)$$

We turn our attention to the second equation in (5.11). Using the expression for $F_3(\varepsilon)$ and $F_3(\zeta)$ defined in (5.5) we find that

$$Z_0 = \frac{f_0}{\kappa} \frac{1}{X_3{}^2 - \frac{C_I{}^2}{C_L{}^2}} E_0 \qquad (5.16)$$

For the small scale using (2.21) we find that

$$Z_0 = f_0 \frac{\kappa}{\kappa - 1} H_0{}^2(k^2 + \ell^2) E_0 \qquad (5.17)$$

while (2.24) for the large scale gives

$$Z_0 = \frac{1}{4} f_0 \frac{\kappa}{\kappa - 1} (1 + N^2) E_0 \qquad (5.18)$$

The main result of the analysis is therefore that if the relations (5.12) and (5.16) are imposed on the initial fields we are assured that W_1 and W_2 will vanish and no sound waves will be present in the linear model. The conditions (5.12) and (5.16) can, of course, be imposed since we may imagine that the calculation is carried out in a spectral domain. Using (5.11) or equivalently (5.12) and (5.16) we note that the latter two expressions in (5.10) reduce to

$$\frac{W_3}{W_0} = \frac{1}{2} + \frac{1}{2} \frac{1}{F_3(\zeta)} \frac{Z_0}{W_0}$$

$$\frac{W_4}{W_0} = \frac{1}{2} - \frac{1}{2} \frac{1}{F_3(\zeta)} \frac{Z_0}{W_0} \qquad (5.19)$$

We shall next investigate the hydrostatic assumption as an initial condition. It is then required that

$$\frac{\partial W}{\partial t} = 0 \qquad\qquad (5.20)$$

$$\frac{\partial^2 W}{\partial t^2} = 0$$

The first condition implies the relation

$$\mu = H_0 \frac{d\varepsilon}{dz} \qquad\qquad (5.21)$$

as seen from the third equation in (5.1). We make use of ε from (5.3) and μ from (5.6) and obtain after some calculations:

$$Z_{0,H} = \frac{1}{4} f_0 \frac{\kappa}{\kappa - 1} (1 + N^2) E_{0,H} \qquad\qquad (5.22)$$

(5.22) is the general expression from the hydrostatic equation. A comparison between (5.22) and (5.5) shows that the hydrostatic assumption is a very good approximation on the large scale but breaks down on the small scale. This can be clearly seen from the ratio between Z_0 computed from the exact expressions in(5.5) for $n = 3$ and the hydrostatic expression (5.22). We get:

$$\frac{Z_0}{Z_{0,H}} = 4 \frac{\kappa - 1}{\kappa^2} \frac{1}{1 + N^2} \cdot \frac{1}{X_3^2 - \frac{C_I^2}{C_L^2}} \qquad\qquad (5.23)$$

The ratio is given in Table 5 as a function of wavelength. The second condition in (5.20) leads to Richardsons equation. (5.21) is differentiated with respect to time and we substitute from the system (5.1). For our model we get then a Richardson equation as follows:

$$\frac{\partial^2 W_H}{\partial Z^2} - \frac{1}{H_0}\frac{\partial W_H}{\partial Z} = \frac{\kappa - 1}{\kappa}\left(\frac{D_H}{H_0} - \frac{\partial D_H}{\partial Z}\right) \qquad (5.24)$$

Using (5.2) and the second expression in (5.3) we find that

$$W_{0,H} = \kappa\left(1 - \frac{\kappa - 1}{\kappa^2}\,\frac{4}{1 + N^2}\right) D_{0,H} \qquad (5.25)$$

We may again judge the validity of (5.25) by comparing with the exact relation in (5.5) for n = 3 giving:

$$\frac{W_0}{W_{0,H}} = \kappa^2\,\frac{\left(1 + \frac{C_I^2}{C_L^2} - X_3^2\right)(1 + N^2)}{(2 - \kappa)^2 + \kappa^2 N^2} \qquad (5.26)$$

This ratio is also given in Table 5.

Summarising the results we may say that the solutions in (5.10) are such that we can see immediately how we can reduce the amplitudes of the sound waves to zero leading to the relations (5.11). These relations are, however, equivalent to the procedure of normal mode initialisation because we could just as well have used (5.10) to calculate from the initial data the amplitude W_1 and W_2 and from those the amplitudes D_1, D_2, E_1, E_2, Z_1 and Z_2. The next step is then to adjust the initial data W_0, D_0, E_0 and Z_0 by subtracting the amplitudes related to the sound waves resulting in an adjusted set of initial data: $W_0 - W_1 - W_2$, $D_0 - D_1 - D_2$, $E_0 - E_1 - E_2$ and $Z_0 - Z_1 - Z_2$. Using this data set as initial data will result in vanishing amplitudes of the sound waves in an integration of the linear equations (5.1).

We note also that if we impose the hydrostatic assumption we find relations which agree with the initialisation procedures for sufficiently large scales because (5.18) is identical to (5.22) and (5.15) to (5.25). However, the hydrostatic equation imposed on the initial data is not sufficient to reduce the amplitudes of the sound waves to zero.

6. Numerical Examples

Two examples will be presented in this section. In each example we have computed the partitioning of the initial amplitudes among the four components for each of the parameters W, D, E and Z.

In the first example we have assumed that the initial data is such that the relations (5.11) are satisfied, i.e.

$$D_0 = F_3(D)W_0$$

$$E_0 = -\frac{F_3(\varepsilon)}{F_3(\zeta)} Z_0 \qquad (6.1)$$

The selected values of W_0 and Z_0 were $0.01.ms^{-1}$ and $0.15.ms^{-1}$. The first corresponds to a maximum vertical velocity of 1 cm s^{-1} while the second gives an order of magnitude of $10^{-5}s^{-1}$ for the vorticity amplitude because $Z_0 \cdot 2H_0$ is the factor on the vorticity. The choice of constant values of W_0 and Z_0 for all horizontal scales is naturally unrealistic but we are interested only in demonstrating the technique. The calculations verified that the amplitudes of the sound wave components were zero for all parameters, i.e. $W_1 = W_2 = D_1 = D_2 = E_1 = E_2 = Z_1 = Z_2 = 0$. The equations (5.19) apply in this case. It is easy to show that for this case we have $W_3/W_0 = D_3/D_0$, $W_4/W_0 = D_4/D_0$ and $E_3/E_0 = Z_3/Z_0$, $E_4/E_0 = Z_4/Z_0$. It is thus sufficient to show the results for the vertical velocity and the vorticity. They are given in Table 6 which shows that very large vertical velocities will exist in the remaining gravity-inertia waves on the small scale, while the vorticity amplitude will be divided almost equally between the two waves. The large vertical velocities are due to the large value of the ratio H/L.

The initial field will therefore be divided in two cells with frequencies τ_3 and $-\tau_3$ and non dimensional velocities X_3 and $-X_3$. The first of these will have ascending and the other descending motion, but they will have the same kind of vorticity and pressure perturbations.

The next example is calculated from hydrostatic initial conditions. This means that the following relations hold for the initial state:

$$D_0 = \frac{1 + N^2}{(\frac{2-\kappa}{\kappa})^2 + \kappa N^2} W_0$$

$$E_0 = \frac{4}{f_0} \frac{\kappa - 1}{\kappa} \frac{1}{1 + N^2} Z_0 \qquad (6.2)$$

$W_0 = 0.01 ms^{-1}$ and $Z_0 = 0.15\ ms^{-1}$ are used again. We note from (6.2) that D_0 and E_0 are constants. With the adopted parameters we find $D_0 = 0.089\ ms^{-1}$ and $E_0 = 390.9m$. Note that these values have to be divided by $2H_0$ to indicate the perturbation amplitude. Table 7 for the vertical velocity shows that the amplitude of the sound waves is considerable although W_1 and W_2 tend to be of equal numerical value, but of opposite sign. This fact is very important because it means that only minor adjustments are needed in the initial data to eliminate the sound waves. Theadjusted data for W_0 is $W_0-W_1-W_2$. The percentage change is thus

$$-\left(\frac{W_1 + W_2}{W_0}\right) \cdot 100\% \qquad (6.3)$$

This quantity has been included in Table 7 showing that the change is 6% on small scales decreasing to 1.5% on the large scale. Note also that W_3 and W_4 for the large scale are almost identical for the two data sets in Tables 6 and 7. Tables similar to Table 7 have been computed for D, E and Z. They are not reproduced here except for the percentage change of D given as the last column in Table 7.

7. Concluding Remarks

The present investigation can be considered as a generalisation of Herbert's (1971) work because we have included the Coriolis effect. It is shown that non-hydrostatic effects are very important if a numerical integration shall include atmospheric phenomena on a very small scale.

Another major result is the determination of the initial conditions necessary to guarantee that no sound waves will occur in a linear integration of the non-hydrostatic equations. These conditions deviate from the hydrostatic initial conditions especially on the small scale, indicating, therefore, that hydrostatic initial conditions will not exclude sound waves from an integration.

R E F E R E N C E S

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TABLE 1

Comparison between the approximate speed of sound waves, X_A, computed from (2.26), the exact speed X_E computed from (2.15) and X_S computed from (2.18). Parameters: H_o = 7321m, H = 25000m, $C_L{}^2$ = 0.1 x $10^6 m^2 s^{-2}$, f_o = 10^{-4} S^{-1} and k = ℓ = $2\Pi_L{}^{-1}$(m = 1).

L	X_A	X_E	X_S
0.1	1.61	1.86	1.73
0.2	3.22	3.35	3.00
0.3	4.83	4.91	4.36
0.4	6.44	6.50	5.74
0.5	8.05	8.10	7.14
0.6	9.66	9.70	8.54
0.7	11.27	11.30	9.95
0.8	12.88	12.91	11.36
0.9	14.48	14.51	12.77
1.0	16.09	16.12	14.18
2.0	32.19	32.20	28.30
3.0	48.28	48.30	42.44
4.0	64.38	64.39	56.58
5.0	80.47	80.48	70.72
6.0	96.56	96.58	84.86
7.0	112.66	112.67	99.00
8.0	128.75	128.77	113.14
9.0	144.84	144.86	127.28
10.0	160.94	160.96	141.42

TABLE 2

Comparison between the approximate speed of gravity-inertial waves, X_A computed from (2.25) or (2.16), and the exact speed X_E computed from (2.15). X_{HE} is the speed obtained if $f_o = 0$ in (2.15). Parameters as in Table 1.

L	X_A	X_E	X_{HE}
0.1	0.4314	0.3738	0.3738
0.2	0.4315	0.4152	0.4152
0.3	0.4315	0.4242	0.4240
0.4	0.4316	0.4275	0.4272
0.5	0.4318	0.4291	0.4288
0.6	0.4319	0.4301	0.4292
0.7	0.4321	0.4308	0.4301
0.8	0.4323	0.4313	0.4304
0.9	0.4326	0.4318	0.4306
1.0	0.4329	0.4322	0.4308
2.0	0.4372	0.4371	0.4313
3.0	0.4444	0.4443	0.4314
4.0	0.4542	0.4542	0.4314
5.0	0.4665	0.4665	0.4314
6.0	0.4811	0.4812	0.4314
7.0	0.4979	0.4979	0.4314
8.0	0.5165	0.5166	0.4314
9.0	0.5369	0.5369	0.4314
10.0	0.5587	0.5588	0.4314

TABLE 3

Comparison of approximate speeds, obtained from (2.21), and exact speeds, obtained from (2.15), for sound waves.

L	X_A	X_E
0.01	0.9976	1.0105
0.02	0.9903	1.0420
0.03	0.9781	1.0940
0.04	0.9606	1.1652
0.05	0.9378	1.2534
0.06	0.9090	1.3558
0.07	0.8739	1.4697
0.08	0.8314	1.5926
0.09	0.7806	1.7225
0.10	0.7196	1.8577

TABLE 4

Comparison of approximate speeds, obtained from (2.21), exact speeds, from (2.15), and speeds based on the hydrostatic assumption for gravity waves.

L	X_A	X_E	X_H
0.01	0.0694	0.0687	0.4314
0.02	0.1389	0.1333	0.4314
0.03	0.2083	0.1904	0.4314
0.04	0.2778	0.2384	0.4315
0.05	0.3472	0.2770	0.4315
0.06	0.4167	0.3073	0.4315
0.07	0.4861	0.3308	0.4315
0.08	0.5556	0.3488	0.4315
0.09	0.6250	0.3629	0.4315
0.10	0.6944	0.3738	0.4315

TABLE 5

The ratios of the true initial condition and the hydrostatic initial condition as a function of wave-length.

L (10^6m)	$\frac{Z_0}{Z_{0,H}}$	$\frac{W_0}{W_{0,H}}$
0.01	20.12	1.24
0.02	5.67	1.19
0.03	3.01	1.15
0.04	2.10	1.12
0.05	1.69	1.09
0.06	1.47	1.07
0.07	1.34	1.06
0.08	1.26	1.05
0.09	1.20	1.04
0.1	1.16	1.03
0.2	1.04	1.01
0.3	1.02	1.00
0.4	1.01	1.00
0.5	1.01	1.00
0.6	1.00	1.00
0.7	1.00	1.00
0.8	1.00	1.00
0.9	1.00	1.00
1.0	1.00	1.00

TABLE 6

Vertical velocities and vorticities as a function of wavelength (see 6.1).

L	W_3	W_4	Z_3	Z_4
0.01	19.93	-19.92	0.075	0.075
0.02	18.33	-18.32	0.075	0.075
0.03	16.26	-16.25	0.075	0.075
0.04	14.19	-14.18	0.075	0.075
0.05	12.36	-12.35	0.075	0.075
0.06	10.84	-10.83	0.075	0.075
0.07	9.59	-9.58	0.075	0.075
0.08	8.57	-8.56	0.075	0.075
0.09	7.73	-7.72	0.075	0.075
0.1	7.03	-7.02	0.075	0.075
0.2	3.64	-3.63	0.075	0.075
0.3	2.44	-2.43	0.075	0.075
0.4	1.84	-1.83	0.075	0.075
0.5	1.47	-1.46	0.075	0.075
0.6	1.23	-1.22	0.075	0.075
0.7	1.06	-1.05	0.075	0.075
0.8	0.93	-0.92	0.075	0.075
0.9	0.82	-0.81	0.075	0.075
1.0	0.74	-0.73	0.076	0.074
2.0	0.38	-0.37	0.076	0.074
3.0	0.26	-0.25	0.076	0.074
4.0	0.21	-0.20	0.077	0.073
5.0	0.17	-0.16	0.077	0.073
6.0	0.15	-0.14	0.077	0.073
7.0	0.14	-0.13	0.078	0.072
8.0	0.13	-0.12	0.078	0.072
9.0	0.12	-0.11	0.078	0.072
10.0	0.12	-0.11	0.078	0.072

TABLE 7

Vertical velocities and percentage differencies for vertical velocities and divergencies as a function of wavelength. (See 6.2 and 6.3)

L	W_1	W_2	W_3	W_4	(W*-W)/W %	(D*-D)/D %
0.01	-1.5556	1.5555	16.519	-16.509	1	-20
0.02	-2.6534	2.6531	15.870	-15.860	3	-10
0.03	-3.0943	3.0938	14.756	-14.745	5	-10
0.04	-3.0131	3.0125	13.353	-13.343	6	-6
0.05	-2.6762	2.6756	11.913	-11.902	6	-4
0.06	-2.2787	2.2781	10.595	-10.584	6	-3
0.07	-1.9111	1.9105	9.455	-9.445	6	-1.8
0.08	-1.6002	1.5997	8.493	-8.482	5	-1.2
0.09	-1.3464	1.3459	7.684	-7.674	5	-0.9
0.1	-1.1415	1.1411	7.002	-6.992	4	-0.6
0.2	-0.33123	0.33100	3.637	-3.626	2.3	-0.09
0.3	-0.15150	0.15131	2.442	-2.432	1.9	-0.03
0.4	-0.08616	0.08599	1.838	-1.827	1.7	-0.02
0.5	-0.05549	0.05532	1.473	-1.463	1.7	-0.01
0.6	-0.03871	0.03855	1.230	-1.220	1.6	-0.01
0.7	-0.02855	0.02839	1.056	-1.046	1.6	-0.00
0.8	-0.02194	0.02178	0.9262	-0.9160	1.6	-0.00
0.9	-0.01740	0.01725	0.8249	-0.8147	1.5	-0.00
1.0	-0.01415	0.01400	0.7439	-0.7338	1.5	-0.00
2.0	-0.00374	0.00359	0.3819	-0.3718	1.5	-0.00
3.0	-0.00181	0.00166	0.2643	-0.2541	1.5	-0.00
4.0	-0.00113	0.00098	0.2075	-0.1974	1.5	-0.00
5.0	-0.00082	0.00067	0.1750	-0.1648	1.5	-0.00
6.0	-0.00065	0.00050	0.1543	-0.1442	1.5	-0.00
7.0	-0.00055	0.00040	0.1404	-0.1302	1.5	-0.00
8.0	-0.00048	0.00033	0.1305	-0.1203	1.5	-0.00
9.0	-0.00043	0.00028	0.1232	-0.1131	1.5	-0.00
10.0	-0.00040	0.00025	0.1177	-0.1076	1.5	-0.00

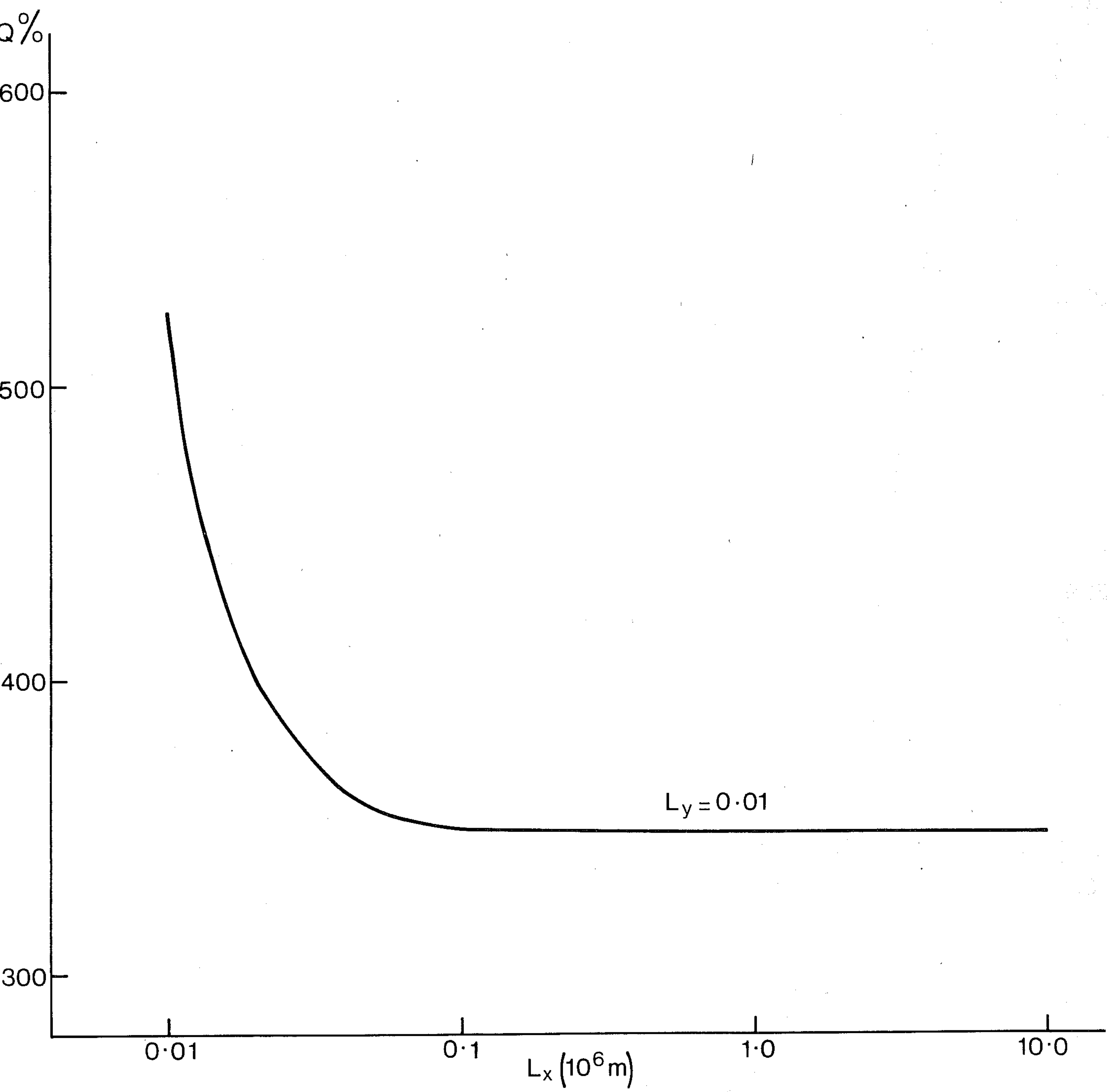

Fig.1 The percentage difference between the exact and the hydrostatic phase speeds as a function of the wavelength in the x-direction when the wavelength in the y-direction is 10km and the vertical wave number is 1.

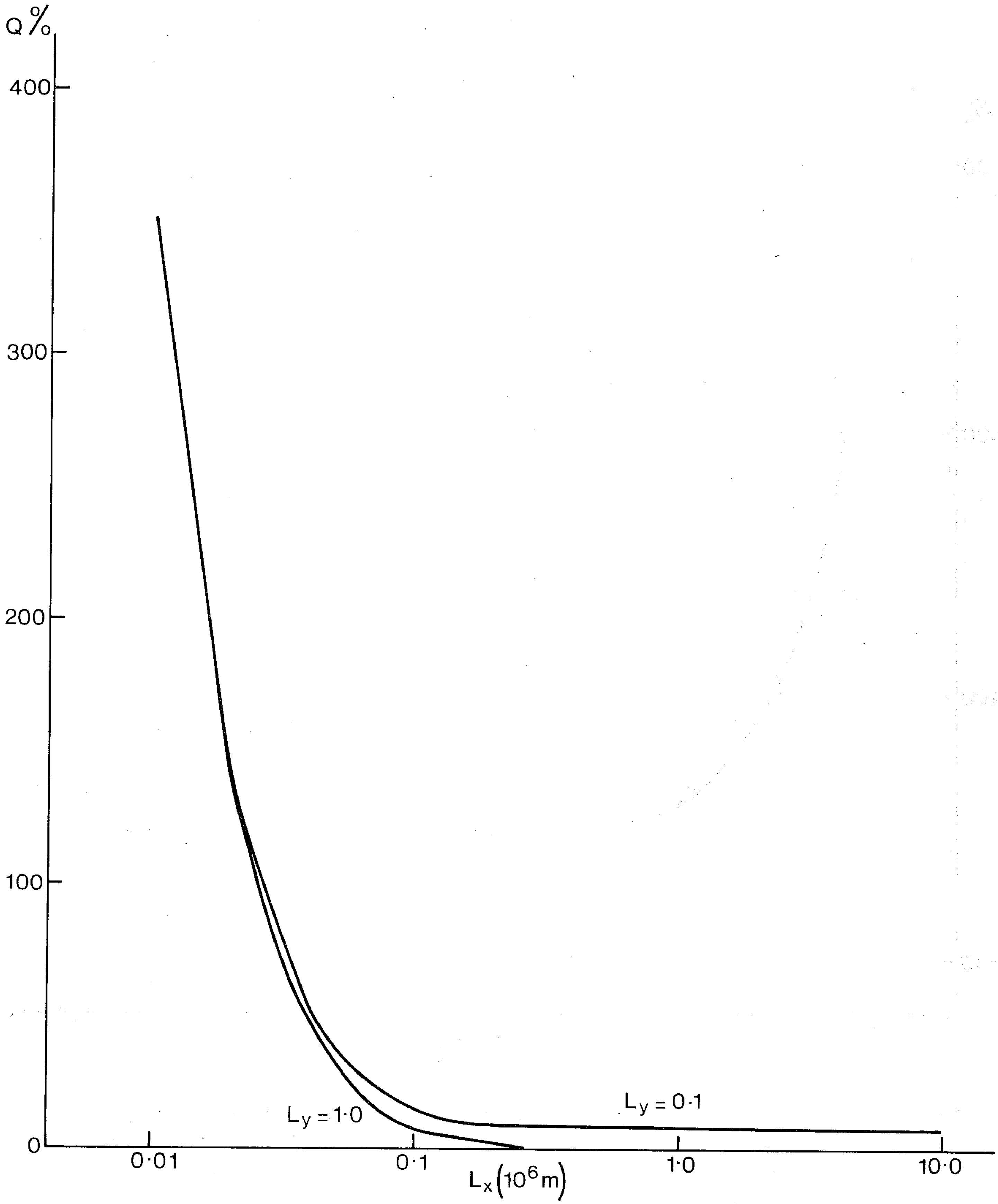

Fig.2 As Fig.1., but the wavelength in the y-direction is 1000km.

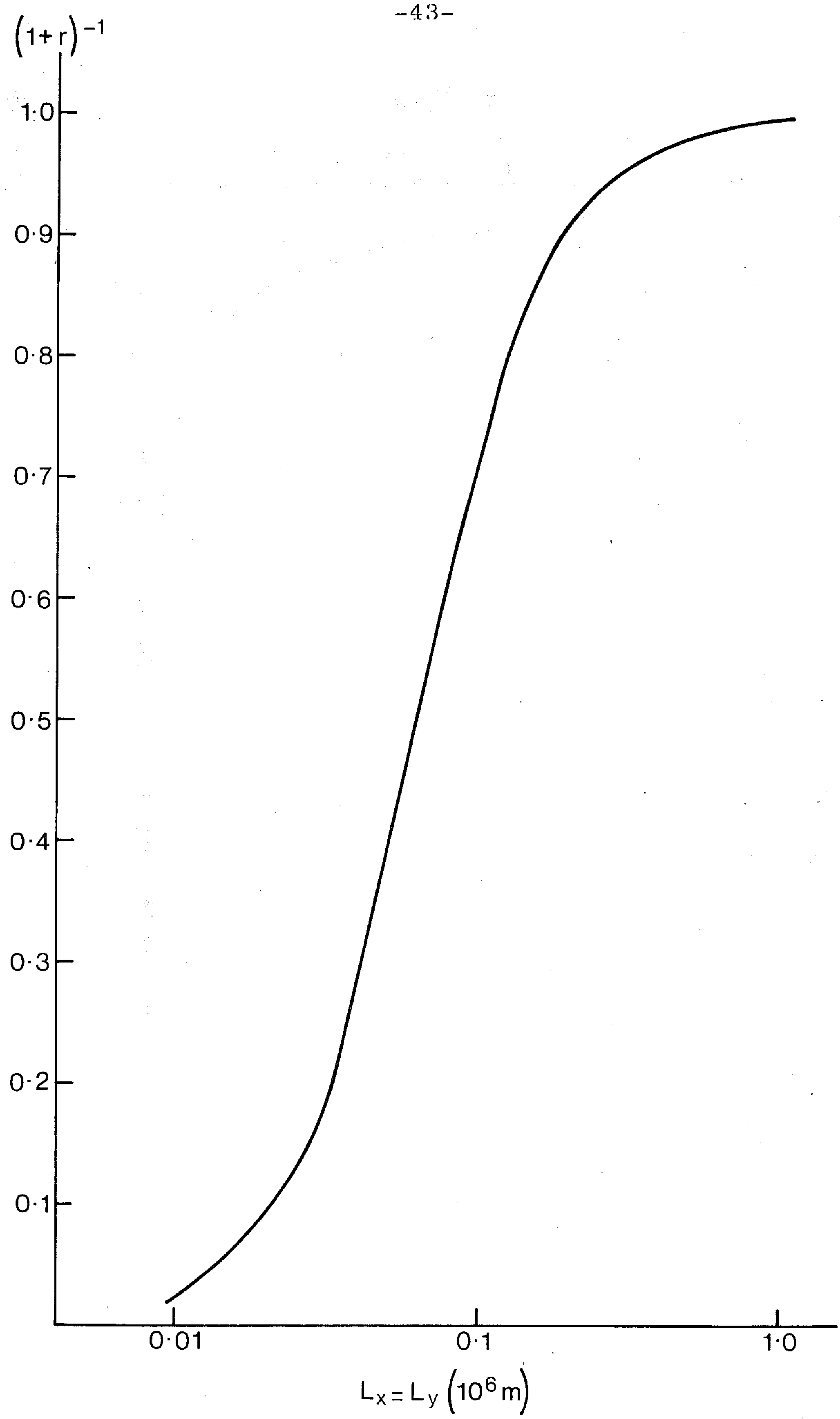

Fig.3 A curve showing the validity of the hydrostatic assumption in an incompressible atmosphere. The calculation is based on the ratio r given in (3.7), and the curve shows $(1 + r)^{-1}$ as a function of wavelength assuming that the wavelengths in the x and y directions are equal.

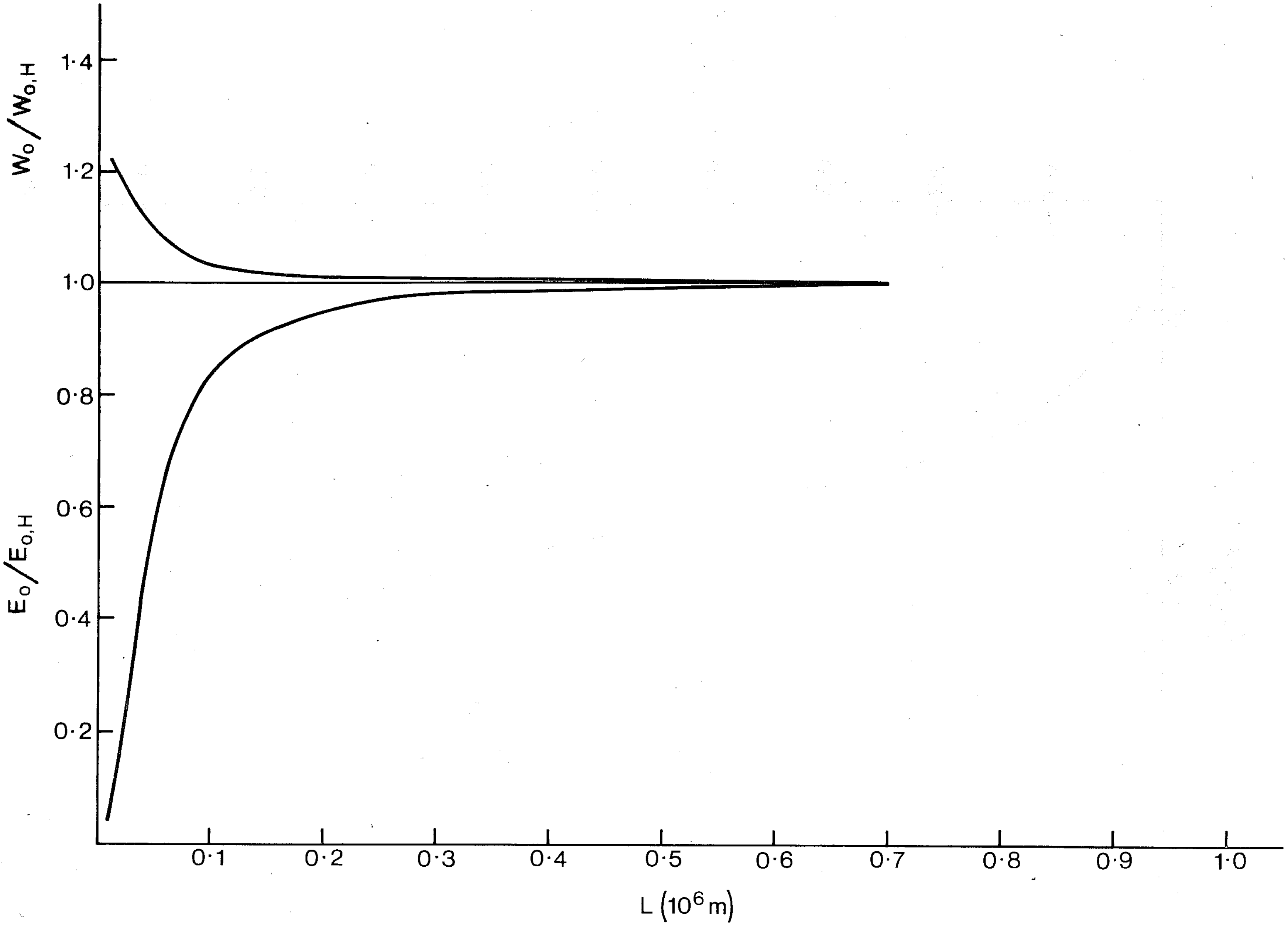

Fig.4 The ratio of the pressure amplitudes in the exact and hydrostatic cases as a function of wavelength, and the ratio between the vertical velocities in the same two cases as a function of wavelength.

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