AN INTRODUCTION TO THE GENERALIZED LAGRANGIAN-MEAN DESCRIPTION OF WAVE, MEAN-FLOW INTERACTION*

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ABSTRACT

The generalized Lagrangian-mean description is motivated and illustrated by means of some simple examples of interactions between waves and mean flows, confining attention for the most part to waves of infinitesimal amplitude. The direct manner in which the theoretical description leads to the wave-action concept and related results, and also to the various 'noninteraction' theorems, more accurately non-acceleration theorems, is brought out as simply as possible. Variational formulations are not needed in the analysis, which uses elementary principles only.

The significance of the generalized Eliassen-Palm relations as conservation equations for wave activity is discussed briefly, as is the significance of the temporal nonuniformity of the generalized Lagrangian-mean description for dissipating disturbances.

1. INTRODUCTION

A topic which has fascinated me for a number of years now (so much so that I have rashly undertaken to write a book on it!) is the interaction of waves and mean flows seen both from a general viewpoint and also in connection with specific applications including those in stratospheric meteorology. Phenomena associated with wave transport processes and nonlinear rectification have long been familiar in simpler contexts such as acoustics – e.g. radiation stress (Brillouin, 1936), acoustic streaming (Lighthill, 1978) – but the subject has been revitalized by recent evidence that wave-induced streaming effects take place on a very large scale in the middle atmosphere* (e.g. Holton, 1975). These do not merely perturb its general circulation, but represent gross features which would otherwise be absent. The clearest example so far documented is the quasi-biennial oscillation of the equatorial zonal wind, hereafter "QBO". Another is the sudden warming. There has also been the realization that wave transport effects might maintain the four-day rotation of Venus' outer atmosphere (Fels & Lindzen, 1974; Plumb, 1975) and probably play a role in the history of the Sun's differential rotation (E.A. Spiegel, personal communication) and in the acceleration of the solar wind (Hollweg, 1978 and refs.).

Quite independently of those developments, there has been a revival in the literature of some long-standing controversies on theoretical aspects of the subject of wave transport, particularly the celebrated 'Abraham-Minkowski controversy' on electromagnetic wave 'momentum' in fluid or solid media. This controversy and its less-publicized relatives in acoustics and geophysical fluid dynamics typify certain misconceptions concerning the <u>generalities</u> of the subject, which despite clarifications now available (e.g. Peierls, 1976) are still widely perpetuated as the scientific literature proliferates with less interdisciplinary communication. (In section 5 we shall catch a glimpse of how these generalities relate to the fluid-dynamical problems which are of more immediate concern to us here.)

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*i.e. the mesosphere and stratosphere

It has become increasingly evident in recent years that the underlying theoretical structure of the subject becomes immeasurably clearer if one describes wave disturbances in terms of particle displacements about the mean flow, in place of the more usual eddy velocity fields. There are deep reasons for this, associated for instance with the connection between symmetries and conservation relations (e.g. Bretherton, 1979; Andrews & McIntyre, 1978c). How best to define a disturbance-associated particle displacement for arbitrary, finite-amplitude waves on an arbitrary mean flow is not a trivial question; and it is linked to the equally nontrivial question of how to define the notion of Lagrangian-mean flow in a general manner. However, significant progress has recently been made towards answering these questions, as a result of several lines of work traceable back at least as far as that of Frieman & Rotenberg (1960) and Eckart (1963), and developed by Dewar (1970), Bretherton (1971, 1979), who was perhaps the first to see the importance of the ideas for geophysical fluid dynamics, Soward (1972) (in MHD dynamo theory), Grimshaw (1975), and culminating in a very general theory developed and discussed by Andrews & McIntyre (1978b, hereafter AM; see McIntyre (1979) for further discussion). This theory may be called the GLM ('generalized Lagrangian-mean') theory of wave, mean-flow interaction. It draws together a number of threads in the subject which seem unconnected when viewed more conventionally; and in particular it shows where the Eulerian-mean results of Eliassen & Palm (1961), Charney & Drazin (1961), Dickinson (1969), Fels & Lindzen (1975), Plumb (1975), Boyd (1976) and Andrews & McIntyre (1976, 1978a) really come from, and how they are connected with such things as the wave-action concept and the energy-momentum-tensor formalism of theoretical physics (Andrews & McIntyre, 1978c). These general results, and many others, emerge in an analytically very simple way.

On a more practical level, the GLM theory might help us toward a better understanding of the general circulation of the middle atmosphere — one in which the theoretical description of planetary waves and other departures from the zonal-mean state fits more naturally with observations of tracer motions. Some of the ideas involved are discussed by Dunkerton (1978) and by Matsuno (1979); they have important precedents in the work of Riehl & Fultz (1957), Krishnamurti (1961), Danielsen (1968), Mahlman (1969) and others. Our understanding of how to use the theory in this context is far from complete; however, a discussion of some of the technical difficulties yet to be overcome is attempted in McIntyre (1979).

In this review I shall concentrate on some prototypical idealized problems concerning the interaction of waves and mean flows, with a view to bringing out the simplicity and intuitive appeal of the ideas which originally led to the GLM theory. I shall also try to show by example why such problems are intriguing for the fluid dynamicist as well as the meteorologist. The two sub-problems comprising the problem of wave, mean-flow interaction, namely

(i) How the waves create or change the mean flow, and

(ii) How mean-flow profiles react back on the waves,

are both well illustrated by the simple example of two-dimensional internal gravity waves, as was illuminatingly brought out in a recent paper by Plumb (1977). We use this special example in sections 2 and 3 in order to motivate a discussion of the more general case of waves involving Coriolis forces – and the power of the GLM theory in dealing with them – which follows in sections 4 and 5. A closely related topic, discussed briefly in section 6, is the use of 'generalized Eliassen-Palm relations' in sub-problem (ii); this in turn suggests diagnostics which are likely to prove important for interpreting observational data on planetary waves in the middle atmosphere. Finally in section 7 we indicate briefly the ideas involved in the GLM description of finite-amplitude disturbances, and its possible application to the middle atmosphere.

2. TWO-DIMENSIONAL INTERNAL GRAVITY WAVES

One class of problems of special meteorological (and astrophysical) interest is that of 'longitudinally symmetric' mean flow, independent of a coordinate x which is either longitude or its cartesian 'channel' equivalent. A special feature of such problems is that there is no longitudinal mean pressure gradient $\partial\bar{p}/\partial x$; thus the fluid is free to accelerate in the x direction in response to the wave effects (subproblem (i)). The simplest model example is that of two-dimensional internal gravity waves being generated by a slippery, corrugated boundary moving parallel to itself with constant velocity c , as suggested in Fig.1. It is well known that, if the waves are transient or are being dissipated in some layer $\mathcal{L}$ at the top of the picture, then the mean flow accelerates there. The wave-drag force which the boundary exerts on the fluid is not felt at the boundary, as far as the mean flow is concerned; it is felt at $\mathcal{L}$. This illustrates the well-known ability of waves to set up a mean <u>stress</u>, whereby the effect of a mean force (in this case the horizontal force exerted by the boundary) can be transferred over considerable

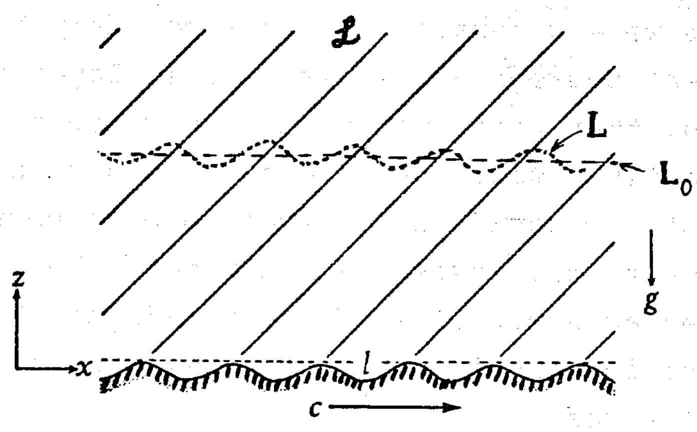

Fig. 1 Internal gravity waves, being generated in a resting, stably-stratified fluid by a rigidly-moving boundary. The sloping lines are lines of constant phase; the disturbance particle paths are parallel to those lines. The buoyancy frequency N is assumed constant. The dotted line L represents an isentropic surface approximately, and L_0 a fixed, horizontal surface.

distances (in this case vertically, up to the layer $\mathcal{L}$).

In this particular case the mean stress involved is simply the Reynolds stress $-\rho\overline{u'w'}$ associated with the waves. Its divergence appears as the wave-induced contribution to the mean-flow acceleration

$$\partial\bar{u}/\partial t = -\partial(\overline{u'w'})/\partial z - \bar{X} , \qquad (2.1a)$$

where $-\bar{X}$ (the sign is chosen for later convenience) is any mean force per unit mass which might also be present (such as a mean viscous force $\propto\partial^2\bar{u}/\partial z^2$). The overbars and primes denote the usual zonal or longitudinal Eulerian average with respect to x, and fluctuations about it. We have assumed a Boussinesq fluid with constant density ρ. Clearly $\overline{u'w'} > 0$ below $\mathcal{L}$ in the picture (the disturbance velocity being directed parallel to the sloping lines of constant phase because the motion is incompressible), and $\overline{u'w'} = 0$ above $\mathcal{L}$ if no disturbance exists there. Then $\partial(\overline{u'w'})/\partial z$ <u>has</u> to be nonzero somewhere in between, which is why the mean flow must accelerate there, apart from any additional effect from $\bar{X}$.

If the waves were generated not at a boundary but by a moving system of heat sources and sinks in some layer in the interior of the fluid, then total momentum integrated over the whole depth of the fluid would have to be constant, and the mean acceleration at $\mathcal{L}$ would be accompanied by a corresponding deceleration where the waves are generated. The latter effect has been proposed as a mechanism for maintaining the fast zonal flow observed in Venus' outer atmosphere (Fels & Lindzen, 1974; Plumb, 1975, & refs.)

If we had used the GLM description instead of the conventional one, the analogue of Eq. (2.1a) would have been

$$\partial\bar{u}^L/\partial t = \rho^{-1}\,\partial(\overline{\zeta_x' p'})/\partial z - \bar{X}^L \qquad (2.1b)$$

for small amplitude, almost-plane waves in the same Boussinesq fluid. (This involves some approximation; the corresponding exact equation is (8.7a) of AM, or rather, the specialization of that equation to two-dimensional motion with constant gravity and zero rotation; see also AM Eq.(8.12).) Here $\overline{(\ )}^L$ denotes a mean along a line of fluid particles distorted by the waves, such as the wavy dotted line L in Fig.1; if the motion were exactly adiabatic, L would exactly coincide with an isentropic surface corrugated by the wave motion. The vertical displacement of the fluid particles about their mean position L_0 is the quantity ζ', whose x-derivative appears, correlated with the disturbance pressure p', on the right of Eq. (2.1b). In section 7 I shall say more precisely how the GLM theory defines $\overline{(\ )}^L$ and the disturbance-associated particle-displacement vector, of which ζ' is the vertical component; the definition will in fact apply to finite-amplitude, arbitrary waves. Note that Eq.(2.1b) makes immediate physical sense; just as Eq.(2. 2.1a) can be obtained by considering the mean stress across fixed, horizontal control surfaces like L_0, so can Eq.(2.1b) be obtained by considering the mean horizontal force $-\overline{\zeta_x p'}$ exerted by the fluid below the wavy, material surface L upon the fluid above it, via the correlation between negative slope $-\zeta_x$ and positive pressure anomaly. (This is exactly the same thing as the wave drag on the boundary itself; the boundary can be regarded as a particular case of a material surface L.) Eq.(2.1b) and its generalizations turn out, as we shall see in section 5, to lead to the easiest way of expressing the connection between mean flow changes and wave dissipation, forcing or transience, in cases more general than that of Fig.1, for instance when a nontrivial mean flow $\bar{u}(z)$ is present. The connection is then a good deal less obvious. But first we digress to look a little more closely at some specific phenomena described by Eqns.(2.1a) and (2.1b). These show in yet another way that there is more to the subject of wave, mean-flow interaction than meets the eye!

3. THE SIMPLEST EXAMPLE OF VACILLATION DUE TO WAVE, MEAN-FLOW INTERACTION: PLUMB AND MC EWAN'S LABORATORY ANALOGUE OF THE QBO.

If the waves in Fig.1 are dissipating throughout the depth of the fluid, then the height scale D for wave attenuation tends to be proportional to the vertical component w_g of the group velocity. For uniform dissipation the mean flow will initially develop as in Fig.2a $[\bar{u} \propto \exp(-z/D)]$, with the biggest change near the boundary* $z = 0$. But now sub-problem (ii) comes in: the feedback of the mean-flow changes onto the waves affects D.

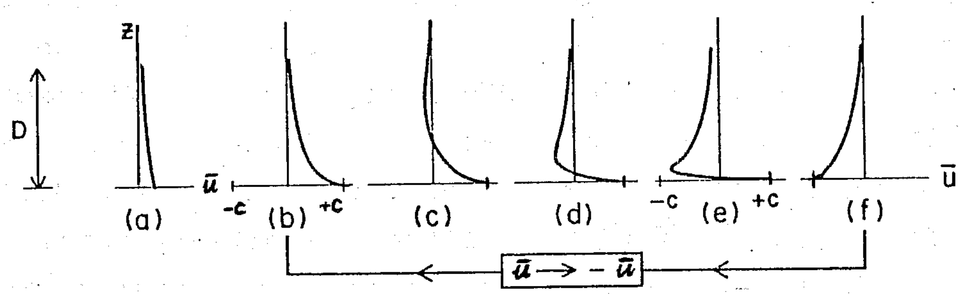

Fig.2 (a),(b): Effect on the mean-flow profile $\bar{u}(z)$ of a single internal gravity wave with phase speed $+c$ at two successive times. (c) - (f): effect of two waves with phase speed $\pm c$, after Plumb,(1977).

When the intrinisc phase speed $c - \bar{u}$ gets small enough, w_g becomes small too (a fact which we shall use again, and which is easily verified from the dispersion properties of plane internal gravity waves); therefore D decreases and also becomes a function of z — we may still speak of it as the local height scale for wave dissipation — and it is smallest of all near $z = 0$. Clearly this cannot go on forever since there is a limiting situation, shown schematically in Fig.2b, in which $\bar{u} = c$ at $z = 0$, and no more waves are generated and no more wave-induced mean-flow change takes place. Actually linear theory must break down near $z = 0$ before this situation is

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*It should be pointed out here that a contrasting situation can occur in the non-Boussinesq case where the density scale height is smaller than D : the greatest mean-velocity change then occurs far above the boundary. This point has been made by Dunkerton (1979a).

reached, but the idea is qualitatively right. We are tacitly assuming that viscosity has a negligible effect on the mean flow, especially near $z = 0$.

If we now add to the input of waves at $z = 0$ a component travelling with equal and opposite phase speed $-c$, something very interesting happens. (The first theory demonstrating the effect was that of Holton & Lindzen (1972), and our understanding of it has been greatly improved by the recent work of Plumb (1977).) Suppose for simplicity that the two waves, with phase speeds $\pm c$, have equal amplitudes so that the boundary is now executing a standing wave

$$z = h(x,t) \equiv a \sin k(x - ct) + a \sin k(x + ct) = 2a \sin kx \cos kct \; ; \qquad (3.1)$$

and suppose moreover that $2kc$ is less that 0.816 times the buoyancy frequency N of the stratification. Then not only can the leftward-travelling component propagate even if $\bar{u} = +c$, but it can also be shown that the relation between w_g and intrinsic horizontal phase speed is strictly monotonic, so that w_g and therefore D for the leftward-travelling component is necessarily larger for all values of z, than it was for the rightward-travelling component before the mean flow developed. Thus, it is easy to see that the leftward-travelling wave will now induce a negative acceleration $\partial\bar{u}/\partial t$ throughout a comparatively deep layer, leading to the appearance of a downward-moving zero in the mean-velocity profile as shown in Figs.2c-e. In Fig.2e, D for the leftward-travelling wave has become small just above the narrow shear layer at the bottom; however, the leftward-travelling wave cannot quite destroy the shear layer by itself because if $\bar{u}$ were to become slightly different from $+c$ at $z = 0$ the effect of the rightward-travelling wave would reassert itself in a very shallow layer near $z = 0$. The shear layer must nevertheless get destroyed sooner or later, either because mean viscous effects become dominant (Plumb, 1977) or, more likely in a real fluid, because the Richardson number becomes smaller and the shear layer goes turbulent. This will quickly wipe out the shear layer and leave us all of a sudden with something like the profile of Fig.2f – i.e. qualitatively like Fig.2b, but with the sign changed. Plumb (op.cit.) refers to this transition between the profiles of Figs. 2e and 2f as "switching". At this point, we can see that the same sequence of events will take place all over again, with the signs changed. The double feedback loop, sub-problems (i) and (ii), between the mean flow and the dissipating waves, has led to a vacillation cycle in which the mean flow reverses again and again, entirely because of the constant input of waves. Figs. 2b - 2f qualitatively depict just half this vacillation cycle.

That such phenomena unquestionably occur in real fluids has recently been most beautifully demonstrated in the laboratory by Plumb & McEwan (1978). They took an annulus of salt-stratified fluid (not rotating) and introduced a standing wave via the motion of a rubber membrane at the bottom, so that equal amounts of clockwise and anti-clockwise-travelling waves (with periods of a fraction of a minute) were generated. The initial conditions involved no mean flow – an almost completely symmetrical situation – yet sooner or later substantial mean flows would appear, going through a vacillation cycle just as in Figs. 2b - 2f. The initial state is unstable to the vacillation cycle (Plumb,1977). The period of the cycle depended of course on the wave amplitude a, but was typically an hour or so. No mean flow developed if the wave amplitude a was too small, owing to the stabilizing effect of mean viscous forces. Plumb & McEwan (pers. communication) have produced a moving picture dramatically showing the existence of the mean flow evolving just as suggested in Fig.2, together with the constant-amplitude standing wave on the boundary which causes the whole sequence of events.

As is well known by now (e.g. Holton, 1975), there is good evidence that a precisely similar mechanism underlies the quasi-biennial reversal, every 26 months or so, of the zonal mean wind in the equatorial lower stratosphere. (The 26 months (or so!) is thus to do with the amplitudes which the relevant (tropospherically-generated) waves happen to have – and nothing to do with any 'obvious' periodicity such as the annual cycle as was thought at one time.) Plumb & McEwan's finding that the mechanism is rather easily killed off by viscous diffusion of the mean flow immediately suggests one of the reasons why general circulation models, which tend to have rather large artificial viscosities (as well as resolutions too coarse to describe the waves very well), have not yet succeeded in producing a QBO.

Other, less simple, examples of vacillation cycles due to wave, mean-flow interaction have recently been noted by Holton & Mass (1976) and Holton & Dunkerton (1978). The waves involved are extratropical planetary waves. In those examples, which model aspects of the behaviour of the wintertime stratosphere in high latitudes, wave transience plays an important role in the vacillation dynamics.

4. WAVES INVOLVING CORIOLIS FORCES.

The waves involved in the equatorial QBO – mainly the equatorial planetary waves, but possibly stationary planetary waves from mid-latitudes as well (Andrews & McIntyre, 1976a, §11b and refs.) – involve Coriolis forces and are structurally more complicated than pure internal gravity waves. It is characteristic of such problems that a description of wave-induced momentum transfer of the type given by Eq.(2.1b) – a Lagrangian-mean description – turns out to be more direct than that given by Eq.(2.1a). The Lagrangian-mean description gives by far the simplest route, for instance, to computing the striking effect of different wave dissipation mechanisms on the latitudinal profiles of $\partial\bar{u}/\partial t$ for equatorial waves (Andrews & McIntyre, *op. cit.*), an effect which is really quite arduous to compute, even to leading order, by conventional methods (*ibid.*, 1976b). Other examples where computations of wave-induced mean effects simplify in a similarly drastic manner have been given by

Grimshaw (1979) and Leibovich (1979) – the latter concerning a possible mechanism for generating Langmuir vortices in the oceanic mixed layer.

I shall not repeat the analysis for equatorial waves here, since three different versions are already given elsewhere (in the papers just cited, together with AM §9). Rather, I want to illustrate and compare both types of description (Eulerian-mean and Lagrangian-mean) by means of the simplest relevant problem, namely that of Fig.1, but with a Coriolis force added whose x,y and z components are $(2\Omega v, -2\Omega u, 0)$ when the velocity components are u,v,w): Ω is assumed constant for the moment. In the Boussinesq approximation the linearized disturbance equations may be written, now allowing for three-dimensional motion, and setting the constant reference density ρ equal to unity, as

$$D_t u' + (\overline{u}_y - 2\Omega)v' + \overline{u}_z w' + p'_x = -X' \quad (4.1)$$

$$D_t v' + 2\Omega u' + p'_y = -Y' \quad (4.2)$$

$$D_t w' - \theta' + p'_z = -Z' \quad (4.3)$$

$$D_t \theta' + \overline{\theta}_y v' + \overline{\theta}_z w' = -Q' \quad (4.4)$$

$$u'_x + v'_y + w'_z = 0 \;, \quad (4.5)$$

where θ is the buoyancy acceleration, $D_t \equiv \partial/\partial t + \overline{u}\partial/\partial x$ the rate of change following the mean flow, the latter being assumed to be of the form $\{\overline{u}(y,z),0,0\} + O(a^2)$, i.e. directed almost parallel to x . Just as we allowed for a mean viscous or other 'extra' force $-\overline{X}$ in Eq.(2.1a), here we permit a correspondingly arbitrary $O(a)$ force $-X'$ on the right of Eq.(4.1), together with corresponding components Y' and Z' in Eqs.(4.2), (4.3), and an arbitrary heating rate $-Q'$ in the buoyancy Eq. (4.4). As with all other primed quantities we have $\overline{X'} = \overline{Y'} = \overline{Z'} = \overline{Q'} = 0$. The buoyancy frequency N is equal to

$\overline{\theta}_z^{\frac{1}{2}}$ in the present notation.

To obtain a physically well-posed problem for the mean flow it is simplest to suppose that the flow is bounded laterally by a pair of vertical walls $y = 0$, b on which the normal component of velocity vanishes (see Fig.4 below) implying that

$$\overline{v} = \overline{v}^L = 0 \quad \text{on} \quad y = 0, b \; . \tag{4.6}$$

Since the bottom boundary is a rigid surface which is impermeable to the fluid, it is plausible (and in fact true for steady waves) that the Lagrangian-mean vertical velocity is zero there also:

$$\overline{w}^L = 0 \quad \text{on} \quad z = 0 \; . \tag{4.7}$$

We must beware, however, of assuming that the Eulerian-mean velocity $\overline{w}$ vanishes at $z = 0$; in fact, for a rigidly-translating, corrugated boundary whose shape is described by a given function $z = h'(x - ct, y)$, where $h' = O(a)$, $\overline{h'} = 0$, and c is a (real) constant as before, it can be shown that (for more detail see Andrews, 1979).

$$\overline{w} = \partial(\overline{v'h'})/\partial y + O(a^3) \quad \text{on} \quad z = 0 \; . \tag{4.8}$$

This is one of the ways in which the conventional Eulerian-mean description is more complicated that a Lagrangian-mean description. The Eulerian-mean velocity $\overline{w}$, which is an average along a <u>horizontal</u> line such as ℓ in Fig.3, is associated with a

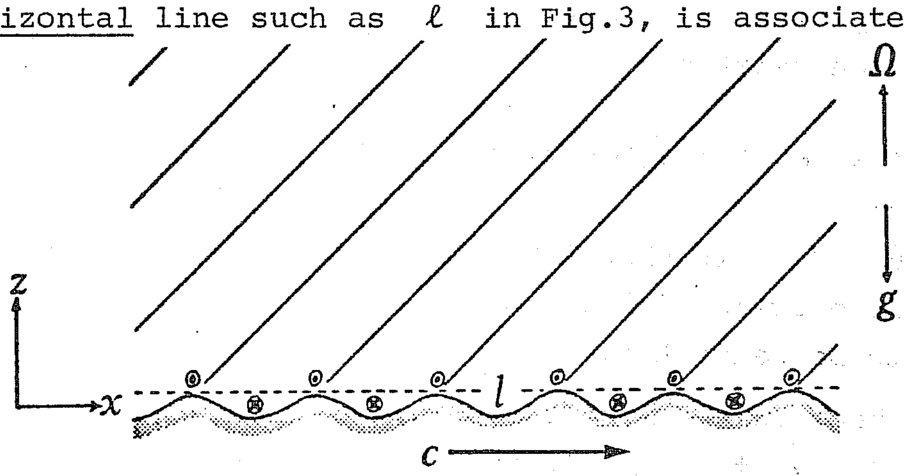

Fig. 3 The reason why $\overline{w} \neq 0$ at $z = 0$ in the rotating problem. (The forward slope of the wave crests is correct when $N^2 (=\overline{\theta}_z) > 4\Omega^2$.)

vertical mass flux, into or out of the thin region between ℓ and the actual boundary. That flux must satisfy continuity with a horizontal, $O(a^2)$ mass flux within that region, associated with any tendency for the disturbance velocity to be one way along valleys and the other way along ridges in the boundary (Fig.3).

In fact, such a tendency turns out to be the rule rather than the exception when Coriolis effects matter; for instance if h' is of the form $a \sin k(x - ct)$ then the disturbance y-velocity v' for conservative, plane inertio-gravity waves on a uniformly stratified basic state of rest turns out to be in quadrature with the z-velocity w' and therefore in phase with h' at $z = 0$ (ignoring signs). This can easily be verified (for further details see Andrews, 1979) by setting $N^2 = \overline{\theta}_z$ = constant, $\overline{u} = \overline{\theta}_y = 0$, and $\underset{\sim}{X}' = 0$, $Q' = 0$, and calculating the structure of elementary plane-wave solutions $\propto \exp i(kx+mz-\omega t)$ of the linearised disturbance Eqs.(4.1) - (4.5). Other pertinent features of such plane-wave solutions are that θ', the disturbance buoyancy acceleration, being proportional to the vertical displacement ζ' through the basic stable stratification $\overline{\theta}_z$, is (like h' at $z = 0$) in quadrature with the vertical velocity w'; also incompressibility dictates that u' is in phase with w' since Eq.(4.5) implies that $iku' + imw' = 0$. Note therefore that $\overline{u'w'}$, $\overline{v'\theta'}$ are nonzero, and $\overline{v'w'}$, $\overline{w'\theta'}$ zero, in a plane inertio-gravity wave. The frequency of such a wave, $\omega(=kc)$, satisfies the well-known dispersion relation

$$\omega^2 = (N^2k^2 + 4\Omega^2m^2)/(k^2 + m^2) \tag{4.9}$$

when $\overline{u} = 0$. (It should be noted that this implies that c^2 must lie between $4\Omega^2/k^2$ and N^2/k^2 for the inertio-gravity waves to be generated.)

Coming back to the mean-flow problem, and still assuming that $\overline{\underset{\sim}{u}} = (\overline{u},0,0) + O(a^2)$, I shall write the mean-flow equations for both the conventional and Lagrangian-mean descriptions correct to $O(a^2)$ next to each other for easy comparison, starting with the counterparts to Eqs.(2.1a) and (2.1b):

$$\partial\underset{*}{\overline{u}}/\partial t + \overline{u}\cdot\nabla\overline{u} - 2\Omega\underset{*}{\overline{v}} + \overline{X} = -(\overline{u'v'})_y - (\underset{*}{\overline{u'w'}})_z \tag{4.10a}$$

$$\partial\underset{*}{\overline{u}^L}/\partial t + \overline{u}^L\cdot\nabla\overline{u}^L - 2\Omega\overline{v}^L + \overline{X}^L = (\overline{\eta'_x p'})_y + \underset{*}{(\overline{\zeta'_x p'})_z} \tag{4.10b}$$

$$\partial\overline{v}/\partial t + 2\Omega\underset{*}{\overline{u}} + \underset{*}{\overline{p}_y} + \overline{Y} = O(a^2) \text{ forcing} \tag{4.11a}$$

$$\partial\overline{v}^L/\partial t + 2\Omega\underset{*}{\overline{u}^L} + \underset{*}{(\overline{p}^L)_y} + \overline{Y}^L = O(a^2) \text{ forcing} \tag{4.11b}$$

$$\partial\overline{w}/\partial t \quad -\underset{*}{\overline{\theta}} + \underset{*}{\overline{p}_z} + \overline{Z} = O(a^2) \text{ forcing} \tag{4.12a}$$

$$\partial\overline{w}^L/\partial t \quad -\underset{*}{\overline{\theta}^L} + \underset{*}{(\overline{p}^L)_z} + \overline{Z}^L = O(a^2) \text{ forcing} \tag{4.12b}$$

$$\partial\overline{\theta}/\partial t + \overline{v}\,\overline{\theta}_y + \underset{*}{\overline{w}}\,\overline{\theta}_z + \overline{Q} = -(\underset{*}{\overline{v'\theta'}})_y - (\overline{w'\theta'})_z \tag{4.13a}$$

$$\partial\overline{\theta}^L/\partial t + \overline{v}^L(\overline{\theta}^L)_y + \underset{*}{\overline{w}^L}(\overline{\theta}^L)_z + \overline{Q}^L = \text{ZERO} \tag{4.13b}$$

$$\underset{*}{\overline{v}_y} + \underset{*}{\overline{w}_z} = 0 \tag{4.14a}$$

$$\underset{*}{(\overline{v}^L)_y} + \underset{*}{(\overline{w}^L)_z} = \frac{\partial}{\partial t}\left[\tfrac{1}{2}(\overline{\eta'^2})_{yy} + (\overline{\eta'\zeta'})_{yz} + \tfrac{1}{2}(\overline{\zeta'^2})_{zz}\right]. \tag{4.14b}$$

The explicit form of the $O(a^2)$ forcing on the right of Eqs. (4.11) and (4.12) will not be needed. The terms distinguished by asterisks are those which survive in the case of almost-plane inertio-gravity waves on a mean flow that is sufficiently slowly-varying in space and time. This turns out to imply that the mean flow is approximately geostrophic and hydrostatic, and in particular that the forcing terms on the right of Eqs.(4.11) and (4.12) can be neglected (Andrews, 1979). Time scales for the mean flow must be long compared with <u>both</u> $(\overline{\theta}_z)^{-\frac{1}{2}}$ and $(2\Omega)^{-1}$. The vertical particle displacement ζ' appears as before, and there is now also a particle displacement η' in the y-direction; Eq.(4.10b) can be seen at once to be a plausible generalization of Eq.(2.1b). The Eulerian-mean "a" equations are familiar and require no comment; the Lagrangian-mean "b" equations are derived succinctly in AM§9*; alternatively it is straightforward (although somewhat tedious) to derive them from the Eulerian-mean or "a" equations by applying the formula for the "Stokes corrections" which are defined as $\overline{u}^S \equiv \overline{u}^L - \overline{u}$, $\overline{p}^S \equiv \overline{p}^L - \overline{p}$, etc., and are given correct to $0(a^2)$ by

$$\overline{u}^S = \overline{\xi' \cdot \nabla u'} + \tfrac{1}{2}\overline{\eta'^2}\,\overline{u}_{yy} + \overline{\eta'\zeta'}\,\overline{u}_{yz} + \tfrac{1}{2}\overline{\zeta'^2}\,\overline{u}_{zz} \, , \qquad (4.15)$$

and similarly for $\overline{p}^S$ or any other Stokes correction, where $\underset{\sim}{\xi}' = (\xi,\eta,\zeta)$ is the vector particle displacement defined in §5 below. Note that the Stokes corrections are $0(a^2)$ <u>wave properties</u> (i.e. can be evaluated to $0(a^2)$ from a lineraized wave solution). As explained further in McIntyre (1979,§3), the terms in Eq.(4.15) involving the second derivatives of $\overline{u}$ are not usually mentioned in classical accounts of Stokes corrections; note that in evaluating those terms it is immaterial whether $\overline{u}$ or $\overline{u}^L$ $(=\overline{u} + 0(a^2))$ appears. Note also that, for consistency with our assumptions that $\overline{v}$ and $\overline{w}$ are $0(a^2)$ (and thus $\overline{v}^L$ and $\overline{w}^L$ also) it is expedient to assume also that $\overline{X},\overline{Y},\overline{Z}$ and $\overline{Q}$ are $0(a^2)$ (and thus $\overline{X}^L$, $\overline{Y}^L$ and $\overline{Q}^L$ also).

A particularly crucial difference between the two descriptions of mean-flow evolution is the difference between the right-hand sides of Eqs.(4.13a) and (4.13b). The Lagrangian-mean equation (4.13b) has <u>zero</u> wave-induced forcing on the right; and this, incidentally, remains exactly true at finite amplitude. For adiabatic motion $(Q = 0)$ the Lagrangian-mean description says very naturally that the mean buoyancy field $\overline{\theta}^L(y,z,t)$ is simply advected by the Lagrangian-mean flow. This is not so in the Eulerian-mean description; the 'eddy heat flux' terms on

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*Eqs. (4.10b) - (4.13b) are simply the result of applying the operator $\overline{(\;)}^L$ to the corresponding equations for the total flow, using Eq.(4.15) with ∇p in place of $\underset{\sim}{u}$, and using the basic (exact) result that $\overline{(\partial\phi/\partial t + \underset{\sim}{u}\cdot\nabla\phi)}^L = \partial\overline{\phi}^L/\partial t + \overline{\underset{\sim}{u}}^L\cdot\nabla\overline{\phi}^L$ for any $\phi(\underset{\sim}{x},t)$ (AM Eq.(2.15)); see also McIntyre, 1979,§3.

the right of Eq.(4.13a) are not generally zero. So even when the motion is completely adiabatic the equations say (if we are using the Eulerian-mean description) that the mean flow feels a wave-induced heating or cooling!

The simplest illustration of this artificiality of the Eulerian-mean description is our model example, in which the waves are supposed to have propagated upwards as far as the layer $\mathcal{L}$ in Fig.4, either because they are being dissipated there, or

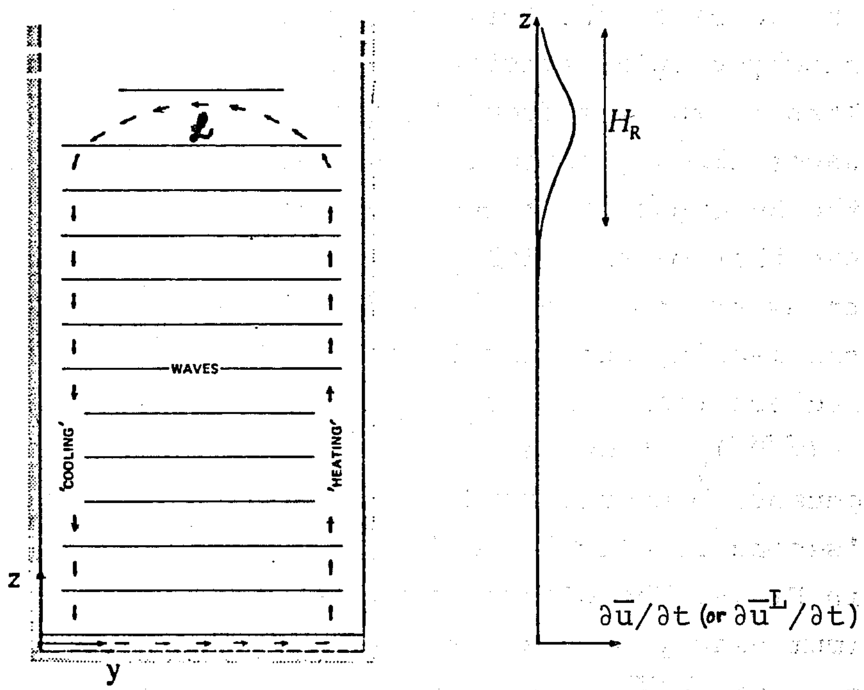

Fig.4 Left: end view (looking along the x-axis) of the inertio-gravity-wave problem. Right: typical profile of the mean acceleration in the longitudinal or x-direction. The left-hand picture indicates how the Eulerian-mean meridional circulation $\bar{v},\bar{w}$ is closed by a mass flux 'within' the bottom boundary, as suggested in Fig.3. H_R is the Rossby height $\Omega b/N$. The time scale for mean-flow changes is assumed much longer than both $1/\Omega$ and $1/N$. The longitudinal Stokes drift $\bar{u}^L - \bar{u}$ is negligible in this problem, but not the meridional and vertical Stokes drift.

because a finite time has elapsed since the bottom boundary started moving. Well below $\mathcal{L}$ we shall take the waves to have reached a steady state and the motion to be conservative, as we originally did in section 2 — we assume that $\underset{\sim}{X}'$ and Q' are zero there, and also $\bar{\underset{\sim}{X}}$ and $\bar{Q}$. To keep life as simple as possible we shall assume that $\bar{u} = 0$ initially, again as in section 2. We also take the buoyancy frequency $N = (\bar{\theta}_z)^{\frac{1}{2}}$ = constant + $O(a^2)$ for the moment. The simplest kind of mathematical analysis for the waves (again for further details see Andrews, 1979) makes the usual kind of 'slowly-varying' approximation, in which the plane-wave solution is assumed locally valid. This involves inter alia an assumption that the layer $\mathcal{L}$ is deep compared with a vertical wavelength and also that $\mathcal{L}$ is deep compared to the Rossby height $H_R = \Omega b/N$. We take h to be of the form $a \cdot f(y) \cdot \sin k(x - ct)$, where $f(y)$ is a slowly-varying function which vanishes at the side wall $y = 0, b$. Then it follows from the properties of plane inertio-gravity waves already noted that the most important term on the right of the x-component of the Eulerian-mean Eq.(4.10a) is $-(\overline{u'w'})_z$ and that on the right of Eq.(4.13a) is $-(\overline{v'\theta'})_y$. The remaining terms are not exactly zero, because plane waves represent only the leading approximation; but in fact it is consistent to neglect them. The response of the mean flow to the forcing $-(\overline{v'\theta'})_y$ together with the forcing represented by the inhomogeneous boundary condition (4.8) involves an Eulerian-mean 'secondary circulation' indicated schematically by the arrows in Fig.4. The picture assumes that the wave amplitude is a maximum near $y = \frac{1}{2}b$ and falls monotonically to zero on either side so that $(\overline{v'\theta'})_y$ changes sign once more, near $y = \frac{1}{2}b$. The mean flow feels an apparent 'heating' on one side of the channel, and 'cooling' on the other. This gives rise to an $O(a^2)$ mean vertical velocity $\bar{w}$ (the asterisked terms in Eq.(4.13a) are in balance, with $Q = 0$); moreover this same $\bar{w}$ just satisfies the boundary condition (4.8). By continuity there must then be a mean motion across the channel, i.e. a contribution to $\bar{v}$, in the vicinity of the layer $\mathcal{L}$ where the wave amplitude goes to zero with height. The Coriolis force associated with this $O(a^2)$ contribution to $\bar{v}$ produces a contribution to $\partial\bar{u}/\partial t$ which is generally comparable with that from the Reynolds stress divergence $-(\overline{u'w'})_z$ in Eq.(4.10a). Thus the 'heating' and 'cooling' on the right of Eq.(4.13a) is important at leading order, in this problem.

The problem for the *Lagrangian-mean* flow is simpler in significant respects; for one thing, there is no Lagrangian-mean flow across any of the boundaries, including the bottom one. This together with the fact that there is also no forcing on the right of Eq.(4.13b) means that the Lagrangian-mean vertical velocity is negligible, sufficiently far below $\mathcal{L}$. In a region of steady waves, when $Q = 0$, the fluid particles merely oscillate about a constant mean level, and have no systematic tendency to migrate up or down. This is no more than might be expected for adiabatic motion in stable stratification; and the Lagrangian-mean description expresses this fact more directly and naturally. Since there is no Lagrangian-mean vertical circulation linking the regions of wave generation and dissipation, and thus no 'Coriolis' contribution to the net wave-induced transport of momentum from one region to the other, the analogue, in the Lagrangian-mean momentum Eq.(4.10b), of the Reynolds stress in the Eulerian-mean momentum Eq.(4.10a), gives a more direct description of the momentum transport. This important fact was recognized intuitively by Bretherton (1969), and the extent to which the result carries over into exact theory is discussed in AM§8.

It has often been assumed in the literature, for instance in connection with thermodynamical, "heat-engine versus refrigerator" arguments, that the nonzero value of $\overline{v'\theta'}$ signifies a tendency for the waves to *transport heat across the channel*. It is clear from the foregoing that while this is true - in the sense that Eq.(4.13a) holds - it is also misleading. For a start, (4.13a) is not the only way of describing the heat budget. But more important than the theoretical description chosen is the fact, deducible by solving the problem in *any* correct description, that there is no tendency at all for the mean temperature actually to rise on one side and fall on the other if we are sufficiently far below $\mathcal{L}$. In the Eulerian-mean description, the adiabatic heating or cooling associated with $\overline{w}$ compensates the divergence of $\overline{v'\theta'}$. This compensation is intrinsic to the nature of the wave motion, as is underlined by the already-mentioned consideration that individual fluid particles are not being heated or cooled below $\mathcal{L}$ because the motion was assumed adiabatic there. In this sense, $\overline{v'\theta'}$ and $\overline{w}$ are purely artifacts of the Eulerian-mean description. (Similarly, there is nothing in the slightest remarkable about $\overline{v'\theta'}$ in the lower stratosphere being 'countergradient'. Rather, as was noted earlier, the sign of $\overline{v'\theta'}$ for wave-like disturbances depends on the phase relations of the disturbance fields, i.e. on the shapes of particle trajectories, which in turn are determined by the wave dynamics and by which way the waves are propagating — not exclusively by local gradients!)

The right-hand half of Fig.4 schematically indicates the profile of the mean acceleration $\partial\overline{u}/\partial t$. If the layer $\mathcal{L}$ is shallower than the Rossby height H_R , then additional contributions $\overline{v}_R, \overline{w}_R$ to the mean meridional circulation arise in a layer of depth H_R centred on $\mathcal{L}$. These adjust the values of $\partial\overline{\theta}/\partial t$ and $\partial\overline{u}/\partial t$ in such a way as to keep the thermal-wind equation satisfied; there is 'room' for such a circulation only in a layer of depth H_R . The vertical integral of $2\Omega\overline{v}_R$ is zero; therefore the vertical integral of $\partial\overline{u}/\partial t$ is unaffected. Further detail concerning the Eulerian-mean problem can be found in McIntyre (1977,§4) and in Andrews (1979). Incidentally, if we were to relax our assumption that mean time scales are long compared with $(2\Omega)^{-1}$, as might sometimes be appropriate in the transient case where the layer $\mathcal{L}$ moves upwards with velocity w_g , then the $\partial/\partial t$ terms would become important in Eqs. (4.11) and the mean response would no longer be confined to within a Rossby height below $\mathcal{L}$. In such a case the response would take the form of a pattern of zonally-symmetric inertio-gravity waves trailing beneath the moving layer $\mathcal{L}$.

One point we have glossed over so far is the role of the right-hand side of the Lagrangian-mean continuity Eq.(4.14b). Being in the form of a time derivative, it is zero for the steady, conservative waves below $\mathcal{L}$; but in any case it turns out to be negligible everywhere in our simple problem*. This is by no means true, however, in all problems of interest: one example is that of equatorial planetary waves. The latitudinal wave-guide structure involved makes some of the derivatives on the right of Eq.(4.14b) important (AM,§9); the same is true of the second derivatives on the right of the expression (4.15) for the Stokes drift, a point to be watched when checking directly that the Eulerian and Lagrangian-mean descriptions do, indeed, give equivalent answers (ibid.). Some further examples when the right-hand side of Eq.(4.14b) is important — we call this the 'divergence effect' — are discussed in McIntyre (1973, 1979) and in Uryu (1979).

A more subtle point is that it is generally necessary for the waves to be conservative as well as steady, in order for the right-hand side of Eq.(4.14b) to vanish exactly. If the waves are being dissipated by radiative heating and cooling, for example, the wavy material line L in Fig.1 must be expected to become gradually less and less related to the shapes of nearby isentropic surfaces. Such temporally nonuniform behaviour, which is in fact characteristic of any Lagrangian description of real fluid motion, can obviously lead to quantities like the right-hand side of (4.14b) differing from zero even for steady waves; therefore if wave dissipation and transience are both strong the theory does not unequivocally distinguish between the two. In section 7 we shall mention one way of overcoming the temporally nonuniform behaviour of $\overline{(\ )}^L$ which may prove to be of some practical importance in studies of planetary waves in the stratosphere; further discussion is given in McIntyre (1979). The temporal nonuniformity reflects an important physical reality, being intimately bound up with the effects of the waves on the apparent large-scale diffusion of chemical tracers about the Lagrangian-mean motion (e.g. Rhines, 1977; Rhines & Holland, 1979; McIntyre, 1979).

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*The assumption that wave amplitude varies slowly in y is used here.

5. CONSERVATION RELATIONS FOR WAVE ACTIVITY: WAVE-ACTION AND ITS RELATIVES.

We are now almost in a position to see the connection between mean-flow evolution and wave dissipation, forcing or transience, in a far more general way than before - not depending on any special approximations or mean-flow profiles. This can be done (to the present accuracy, $O(a^2)$) via either the Eulerian-mean Eqs.(4.10a) - (4.14a) or the Lagrangian-mean Eqs.(4.10b) - (4.14b); but the latter route is a good deal quicker, and appears to be the only feasible route if we want to derive the further generalization to finite amplitude derived and discussed in AM and in McIntyre (1979). Either route depends on a fundamental conservation relation whose derivation and general significance I shall now indicate.

We shall need all three components ξ', η', ζ' of the disturbance particle displacement $\boldsymbol{\xi}(\mathbf{x},t)$. Correct to $O(a)$, they satisfy $\overline{\boldsymbol{\xi}} = 0$ and (with $D_t \equiv \partial/\partial t + \bar{u}\partial/\partial x$ as before)

$$D_t \eta' = v' \tag{5.1a}$$

$$D_t \zeta' = w' \tag{5.1b}$$

$$D_t \xi' = u^{\ell} \equiv u' + \boldsymbol{\xi}\cdot\nabla\bar{u} = u' + \eta'\bar{u}_y + \zeta'\bar{u}_z \ , \tag{5.1c}$$

together with

$$\nabla\cdot\boldsymbol{\xi}' = 0 \quad , \tag{5.2}$$

if the fluid is incompressible. The manipulations to get the conservation equation are quite simple, and reminiscent of those involved in the familiar operation of forming a kinetic energy equation. However, instead of taking the scalar product of the momentum equation with velocity, we multiply it scalarly by $-\partial\boldsymbol{\xi}'/\partial x$ and average (with respect to x).

It is convenient first to recast the linearized momentum Eqs.

(4.1),(4.2),(4.3) in terms of the quantity u^ℓ , which is the longitudinal disturbance velocity for the displaced fluid particle, as evidenced by the correction term $\underset{\sim}{\xi}'\cdot\nabla\bar{u}$ in Eq.(5.1c). We may call u^ℓ the Lagrangian disturbance velocity, correct to $O(a)$. In vector notation with $\underset{\sim}{u}^\ell \equiv (u^\ell, v' w')$ we have, after a little manipulation in which (4.1),(4.2),(4.3) are added to $\underset{\sim}{\xi}'\cdot\nabla$ {(4.10),(4.11),(4.12)} with $O(a^2)$ terms neglected in the latter three equations,

$$D_t\underset{\sim}{u}^\ell + 2\underset{\sim}{\Omega}\times\underset{\sim}{u}^\ell + \hat{\underset{\sim}{z}}\,q' + \nabla p' + \underset{\sim}{\xi}'\cdot\nabla(\nabla\bar{p}) + \underset{\sim}{X}^\ell = 0 \; , \qquad (5.3)$$

where $\hat{\underset{\sim}{z}}$ is a unit vertical vector, and

$$q' \equiv -\theta^\ell \equiv -\theta' - \eta'\bar{\theta}_y - \zeta'\bar{\theta}_z \; . \qquad (5.4a)$$

The quantity q' is a measure of the departure from adiabatic motion, because it is easy to see from Eq.(4.4) and $\underset{\sim}{\xi}'\cdot\nabla$ of (4.13) that it satisfies

$$D_t\, q' = Q^\ell \qquad (5.4b)$$

correct to $O(a)$; we have also defined $\underset{\sim}{X}^\ell \equiv \underset{\sim}{X}' + \underset{\sim}{\xi}'\cdot\nabla\bar{X}$.

Now

$$\overline{-\frac{\partial\xi'}{\partial x}D_t u^\ell} = D_t(\overline{-\frac{\partial\xi'}{\partial x}u^\ell}) + \overline{u^\ell D_t\frac{\partial\xi'}{\partial x}} \; ,$$

and the last term is $\overline{u^\ell\,\partial u^\ell/\partial x}$ by (5.1c) $= \partial(\tfrac{1}{2}\overline{u^{\ell 2}})/\partial x = 0$.

Similarly

$$-\overline{\frac{\partial\eta'}{\partial x}D_t v'} = D_t(\overline{-\frac{\partial\eta'}{\partial x}v'}) \; ,$$

$$-\overline{\frac{\partial\zeta'}{\partial x}D_t w'} = D_t(\overline{-\frac{\partial\zeta'}{\partial x}w'}) \; .$$

Finally

$$-2\overline{\frac{\partial\underset{\sim}{\xi}'}{\partial x}\cdot\underset{\sim}{\Omega}\times\underset{\sim}{u}^\ell} = -2\overline{\frac{\partial\underset{\sim}{\xi}'}{\partial x}\cdot\underset{\sim}{\Omega}\times D_t\underset{\sim}{\xi}'} = \overline{2\underset{\sim}{\xi}'\cdot\underset{\sim}{\Omega}\times(D_t\frac{\partial\underset{\sim}{\xi}'}{\partial x})}$$

(since $\overline{\partial(\underset{\sim}{\xi}'\cdot\underset{\sim}{\Omega}\times D_t\underset{\sim}{\xi}')/\partial x} = 0$)

$$= -2\overline{\left(D_t\frac{\partial \xi'}{\partial x}\right)\cdot \Omega \times \xi'} \;=\; D_t\overline{\left(-\frac{\partial \xi'}{\partial x}\cdot \Omega \times \xi'\right)}$$

(the end result being half the sum of the second and fourth expressions). Therefore (recalling Eq.(5.2)) the result of scalarly multiplying (5.3) by $-\partial\xi'/\partial x$ and averaging with respect to x is simply

$$D_t \mathcal{P} + \nabla\cdot\{-\overline{\frac{\partial \xi'}{\partial x}p'}\} = \overline{\frac{\partial \xi'}{\partial x}\cdot X^{\ell}} + \overline{\frac{\partial \zeta'}{\partial x}q'} , \tag{5.5}$$

correct to $O(a^2)$, where

$$\mathcal{P} \equiv -\overline{\frac{\partial \xi'}{\partial x}\cdot(u^{\ell} + \Omega \times \xi')} . \tag{5.6}$$

Of course D_t can be replaced by $\partial/\partial t$ in Eq. (5.5), $\nabla\cdot\{\ \}$ by $-\partial(\overline{\eta_x'p'})/\partial y - \partial(\overline{\zeta_x'p'})/\partial z$, and $\overline{\xi_x'\cdot\Omega \times \xi'}$ by $-2\Omega\overline{\xi_x'\eta}$ (as in Andrews & McIntyre, 1976a, Eq.(A15)); but I wanted to exhibit both Eq.(5.5), and its derivation, in a form suggestive of generalizations for other kinds of averaging. For instance if $(\overline{\ })$ were a time average we would have scalarly multiplied Eq.(5.3) by $\partial\xi'/\partial t$, and if $(\overline{\ })$ were an ensemble average over some ensemble label α — for instance the phase of the waves if we are using either the 'random phase' idea or the slowly-varying, 'two-timing' idea — then we would have multiplied Eq.(5.3) by $\partial\xi'/\partial\alpha$. In all these cases the foregoing derivation goes through almost word for word. In Andrews & McIntyre (1978c) it is shown that Eq.(5.5), with α in place of x, and with right-hand side zero, reduces to Bretherton & Garrett's (1968) form of the wave-action equation under their assumption of slowly-varying, conservative waves. (This incidentally provides a simple yet general derivation of Bretherton & Garrett's equation which does not depend on using a variational principle — it was previously thought that the variational approach is not only illuminating, but essential).

Thus the wave property $\mathcal{P}$ is closely related to the wave-action. However, since $\mathcal{P}$ itself arises from spatial, rather than ensemble or phase, averaging — so that conservation of p is associated with translational invariance of the mean flow — p should strictly speaking be called the pseudomomentum, following the usage established in solid-state physics*. In the case of slowly-varying waves, it is easy to see (Andrews & McIntyre, 1978c) that p reduces to Bretherton & Garrett's wave-action times the x component of the wavenumber, k.

Coming back now to our problem of wave, mean-flow interaction with its two sub-problems (i) (waves changing mean flow), and (ii) (mean flow influencing waves), we can now see how Eq.(5.5) plays a role in both. For sub-problem (ii) it evidently comprises a useful tool for both calculating and describing the generation, propagation and dissipation of waves in a given mean zonal flow with arbitrary profiles $\bar{u}(y,z)$, $\bar{\theta}(y,z)$. (Clearly one wants for this purpose a measure of wave activity which is conserved when the waves are not being dissipated or generated; one could then draw 'arrow' pictures of the flux, for instance — perhaps superposed on contours of its divergence — and thus get a direct feel for where (in the meridional (yz) domain) the waves are piling up, or being dissipated. By contrast, the divergence of the usual wave-energy flux $\overline{p'v'}$, $\overline{p'w'}$ does not indicate any such thing.)

Second, Eq.(5.5) reveals the basic structure of the mean-flow-evolution sub-problem (i). Putting it together with the Lagrangian-mean-flow Eq.(4.10b) we see at once that the right-

*The distinction between pseudomomentum and momentum, whose conservation is associated with translational invariance of the total problem, mean flow plus waves, has long been recognized in solid-state physics, and has recently proved to be the main key to unravelling the Abraham-Minkowski controversy mentioned in the introduction (Peierls, 1976). Surprising as it may seem, the controversy stemmed partly from a failure to recognize that translational invariance of the propagating medium is logically not the same thing as translational invariance of the total problem. Another source of confusion has been the fact that in certain special examples p turns out to be numerically equal to the mean momentum; it happens that the problem of section 2 is one such example, in the purely transient, conservative case.

hand side of that equation can be written (correct, as always, to $O(a^2)$)

$$\frac{\partial}{\partial t}\mathcal{P} - \overline{\frac{\partial\xi'}{\partial x}\cdot \underset{\sim}{X}^{\ell}} - \overline{\frac{\partial\zeta'}{\partial x}q'} \quad , \qquad (5.7)$$

showing the dependence of the wave-induced forcing upon wave transience, dissipation, or generation (the remarks at the end of section 4 should be borne in mind). The reason why the results come out this way is again to do with the connection between conservation relations and symmetries, as has been clearly brought out by Bretherton (1979) and further discussed in Andrews & McIntyre (1978c). It turns out that the result of putting Eqs.(4.10b) and (5.5) together can, in the case of conservative waves ($\underset{\sim}{X}$, Q, q' all zero), be derived more directly by applying Kelvin's circulation theorem to a wavy line like L in Fig.1. The most general form of this idea appears to be that expressed by Theorem I of AM for finite-amplitude waves.

However, the argument is not complete without considering the complete set of mean-flow equations (after all, if we had naively forgotten about the right-hand side of Eq.(4.13a) in the Eulerian-mean problem – not to mention the boundary condition (4.8) – we would have got a completely wrong answer for the effective transfer of mean momentum from the boundary to the layer $\mathcal{L}$ in Fig.4!). Nevertheless, there is not much more that need be said, because the right-hand side of Eq.(4.13b) is zero ; and so we have to worry only about the right-hand sides of (4.11b),(4.12b), and (4.14b). The last of these is already in the form of a time derivative. The precise form of the forcing in (4.11b) and (4.12b) is not critical, because those two equations enter the problem for the rate of change of $\bar{u}$ and $\bar{\theta}$ only in time-differentiated form (cf. Andrews & McIntyre,1976a, Eqs.(5.7),(5.8)). That is, since we are interested in solving for the mean-flow tendency at a given moment we may regard

$$\{\partial\bar{u}^L/\partial t,\ \ \partial\bar{\theta}^L/\partial t,\ \ \partial\bar{p}^L/\partial t,\ \ \bar{v}^L,\ \bar{w}^L\} \qquad (5.8)$$

as our basic set of dependent variables and take, as our basic complete set of equations, (4.10b), (4.13b) and (4.14b) together with the time derivatives of (4.11b) and (4.12b). The right-hand sides of the latter must necessarily take the form of time derivatives! The foregoing arguments do not depend on any approximations based on almost-plane waves or special mean-flow profiles, and demonstrate rather generally how mean-flow acceleration is linked to wave dissipation, generation, and transience, in the sense that all wave-induced forcing terms are either time derivatives, like the first term in (5.7), or explicitly involve departures from conservative motion, like the second and third terms in (5.7).

A corollary of the analysis is that for steady, conservative waves there is no mean acceleration, as first shown by Charney & Drazin, (1961) for quasi-geostrophic waves; and this is sometimes called a 'noninteraction theorem'. Fundamentally speaking it should really be called a 'nonacceleration theorem', however: there is an interaction inasmuch as the right-hand sides of Eqs. (4.11b) and (4.12b) are not zero when steady waves are present, and this upsets thermal-wind balance statically by $O(a^2)$. Such interactions, while not always negligible in a description of the mean flow correct to $O(a^2)$, are probably not very important in most meteorological examples; but there are other examples, such as the radiation pressure of sound waves in a box (Brillouin, 1925), the acceleration of the solar wind by Alfvén waves (e.g. Hollweg, 1978), and the 'parametric' transmitter in underwater acoustics, (e.g. Moffett, et al., 1971), where analogous interactions are extremely important.

6. CONSERVATION RELATIONS FOR WAVE ACTIVITY: THE GENERALIZED ELIASSEN-PALM RELATION.

The foregoing results (and various others similarly revealing basic structure in the theory of acoustic and surface-gravity waves on nontrivial mean flows; see AM§6) leave one in no doubt that the description in terms of disturbance-associated particle displacements and Lagrangian means is absolutely fundamental from a theoretical point of view. However, it might still be asked whether the basic conservation relation, Eq.(5.5), could be manipulated into a form not involving the disturbance particle displacements; this would be convenient for observational studies of stratospheric wave activity, for instance. The answer appears to be 'no'; however one can find a conservation relation, Eq.(5.5a) of Andrews & McIntyre (1976a), in which the flux, at least, does not depend on the disturbance displacements. The derivation is given in Appendix A of the same reference, in which the present Eq.(5.5) appears as Eq. (A15). The result has the form

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial y}\{\overline{u'v'} - \overline{u}_z \frac{\overline{v'\theta'}}{\overline{\theta}_z}\} + \frac{\partial}{\partial z}\{\overline{u'w'} + (\overline{u}_y - 2\Omega)\frac{\overline{v'\theta'}}{\overline{\theta}_z}\} = D. \quad (6.1)$$

Here A is a wave property equal to $\mathcal{P}$ plus a number of extra terms* involving $\underset{\sim}{\xi}'$, and D is another expression involving $\underset{\sim}{\xi}'$ which, like the right-hand side of the present Eq.(5.5), is zero for conservative motion ($\underset{\sim}{X}$, Q, q' zero). The flux whose divergence appears in this equation does not involve $\underset{\sim}{\xi}'$; it is the fundamental entity arising in the analysis of Eliassen and Palm, (1961). Eq.(6.1) and its generalizations to spherical geometry (Andrews & McIntyre, 1976a, 1978a) may appropriately be called "generalized Eliassen-Palm relations". They play a role in the theory of the Eulerian-mean flow like that of Equation (5.5) for the Lagrangian-mean flow; and like Eq. (5.5) can also be used as conservation relations for wave activity. This dual role, in sub-problems (i) and (ii), of the Eliassen-Palm flux $\{\overline{u'v'} - \overline{u}_z\overline{v'\theta'}/\overline{\theta}_z,\ \overline{u'w'} + (\overline{u}_y - 2\Omega)\overline{v'\theta'}/\overline{\theta}_z\}$, suggests that it should be regarded as a more fundamental diagnostic than the associated 'wave-energy fluxes' also discussed by Eliassen and Palm. Meridional cross-sections of the Eliassen-Palm flux and its divergence calculated from general circulation statistics are presented and

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* A is minus the lengthy expression within heavy square brackets in Eq.(5.5a) of Andrews & McIntyre, (1976a), with nonhydrostatic terms added; see p.2034 of same reference.

discussed by Edmon et al. (1979). One question which such observational cross-sections should help to answer is the question of whether a singular (zero-wind) line for topography-linked planetary waves behaves more like an absorber (as predicted by linear theory and numerical experiment) or a reflector (as predicted by a certain class of idealized nonlinear theories, e.g. Tung (1979)).

Eliassen and Palm pointed out that their flux reduces to $\{\overline{u'v'}, -2\Omega\,\overline{v'\theta'}/\overline{\theta}'_z\}$ for quasi-geostrophic waves, and it is interesting to note that in the same approximation we have

$$A \simeq \overline{\eta'\mathcal{Q}'} + \tfrac{1}{2}\overline{\mathcal{Q}}_y\overline{\eta'^2} - \frac{2\Omega}{\overline{\theta}_z}\,\overline{\eta'_z q'}\ , \qquad (6.2)$$

where $\mathcal{Q}$ is the quasi-geostrophic potential vorticity*. For the conservative case $\mathcal{Q}' = -\overline{\mathcal{Q}}_y\eta'$ and so Eq.(6.2) reduces still further to

$$A \simeq -\tfrac{1}{2}\overline{\mathcal{Q}}_y\overline{\eta'^2}\ . \qquad (6.3)$$

A significant consequence is that the rate of change of density A of wave activity, in the generalized Eliassen-Palm relation, becomes the flux of quasi-geostrophic potential vorticity in the y-direction (Dickinson, 1969; Bretherton & Haidvogel, 1976) because from (6.3) and (5.1a)

$$\partial A/\partial t = -\overline{\mathcal{Q}}_y\,\overline{\eta'v'} = \overline{v'\mathcal{Q}'}\ .$$

7. THE FINITE-AMPLITUDE THEORY AND THE TEMPORAL NONUNIFORMITY

The Lagrangian-mean theory becomes even more powerful when extended to finite amplitude, and leads to what appear to be the most general forms both of the theorems on mean-flow evolution

*An interesting parallel can be found in Eq.(3.3) of McIntyre & Weissman (1978), which applies to the two-dimensional problem of §2 above.

(sub-problem (i)) and of the conservation relation for wave activity (sub-problem (ii)). The most fundamental question is how to define the idea of Lagrangian-mean flow $\bar{\underset{\sim}{u}}^L$ and disturbance particle displacement $\underset{\sim}{\xi}'(\underset{\sim}{x},t)$ at finite amplitude. At first sight there appears to be an infinite number of choices; but it turns out that the following definition is the one which appears to lead to the simplest general theoretical structure for finite-amplitude waves, in some ways just as simple as the $O(a^2)$ Lagrangian-mean theory of sections 4 and 5.

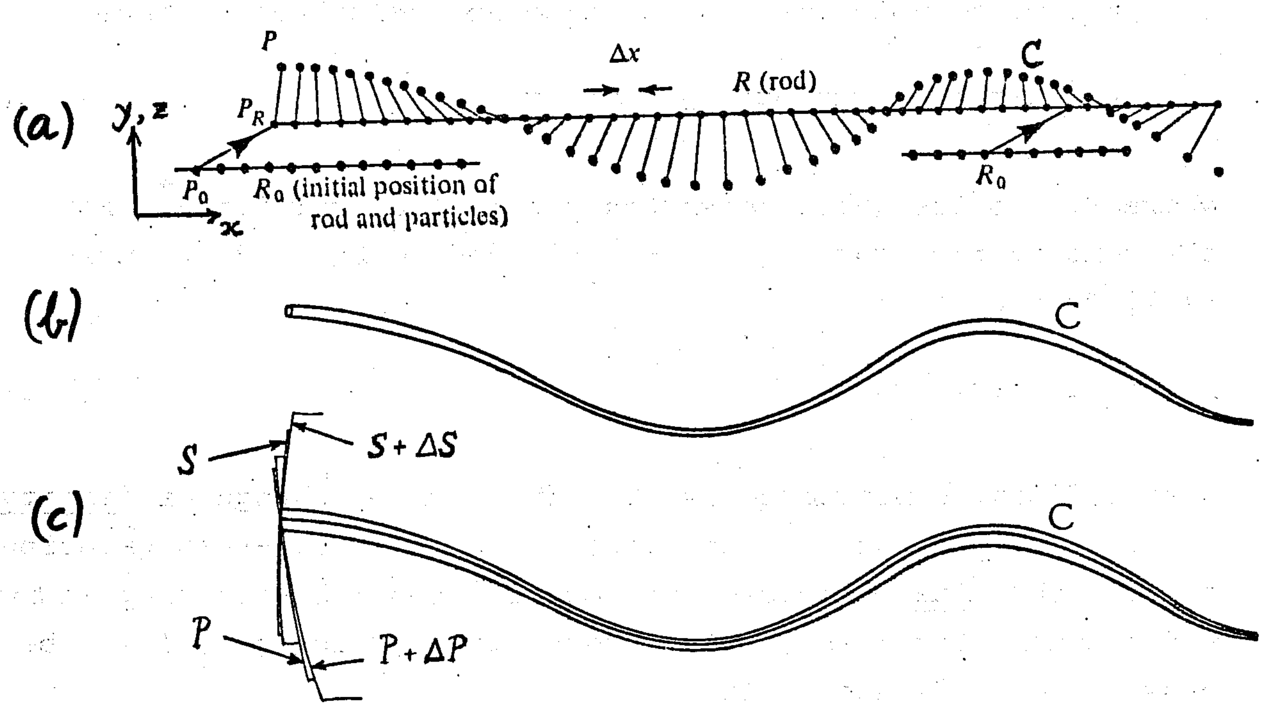

Fig.5 Ways of visualizing the generalized Lagrangian-mean velocity and disturbance particle displacement for disturbances of finite amplitude (see text).

Suppose that there is no disturbance anywhere at some initial time $t = t_0$. In Fig.5a, let R_0 be a line parallel to the x-axis. Fix attention on a row of marked particles which are initially spaced at equal distances Δx along R_0 , and then watch these particles as they follow the fluid motion. We now refer (following Andrews & McIntyre, 1978b) to a mechanical analogy (which has no dynamical connection with the fluid motion) in which we imagine that a thin, light, rigid rod R initially coincides with R_0 , but is subsequently free to move while remaining parallel to the x-axis. The position P of a typical particle of fluid whose initial position was P_0 is joined to the point P_R on R which initially coincided with P_0 ; the ligaments joining the marked particles to R consist of identical 'elastic bands' such that P_R is pulled towards P with a force proportional to the distance P_RP , and similarly for the other points. The rod R is imagined to be in <u>static equilibrium</u> under the pull of all the ligaments. Then, in the limit $\Delta x \to 0$, the velocity with which the rod moves is defined to be $\bar{\underset{\sim}{u}}^L$; and if $\underset{\sim}{x}$ is the current position of P_R, $\underset{\sim}{\xi}'(\underset{\sim}{x},t)$ is defined to be the 'elastic-band vector' $\overrightarrow{P_RP}$.*

It turns out (see Eq.(7.1) below) that $\bar{u}^L$ is exactly equal to the velocity of the <u>centre of mass</u> of a thin tube of fluid initially lying in the x-direction (Fig.5b). This result was conjectured by Matsuno (personal communication) on the basis of a calculation for small disturbance amplitude. A corollary of Matsuno's remark is that the vertical Stokes drift $\bar{w}^L - \bar{w}$ gives a direct measure of the rate of change of disturbance available potential energy; and this too, is given the status of an exact result by the finite-amplitude theory.

The foregoing gives the generalized Lagrangian-mean operator $\overline{(\ )}^L$ corresponding to a spatial (longitudinal or zonal) Eulerian mean; the GLM operators corresponding to the time, ensemble or two-timing varieties of Eulerian averaging operators are defined in AM, where the theory is developed in a form which covers all these cases at once. Once we possess definitions of $\underset{\sim}{\xi}'(\underset{\sim}{x},t)$ and $\bar{\underset{\sim}{u}}^L(\underset{\sim}{x},t)$, we can easily derive finite-amplitude analogues of Eq.(5.5) which are analytically almost as simple, and lead to much the same consequence, as before. (Full details are given in Andrews & McIntyre, 1978c).

*For the corresponding analogy for zonal averaging on the sphere, see McIntyre 1979, §4.

It should be noted that although the operator $\overline{(\ )}^L$ involves averaging along the curve C the average is <u>not</u> uniformly weighted with respect to arc length s along C . The weighting of the average along C can be expressed in terms of the non-uniform thickness of the tube corresponding to C in Fig. 5b, or more accurately in terms of the <u>mass</u> per unit length of the tube, if we want to include the case of fully compressible flow. This is because mass per unit length of C in Fig.5b is just proportional to number of particles per unit length of C in Fig.5a, in the limit $\Delta x \to 0$. Thus if dV is an element of volume of the tube C , so that ρdV is an element of mass, then the GLM of any quantity $\phi(x,t)$ may be defined as

$$\overline{\phi}^L = \int_C \phi\rho dV \Big/ \int_C \rho dV \ , \tag{7.1}$$

in the limit of small tube cross-section (AM§4.3). As Matsuno (1979) and McIntyre (1979) both explain, the nonuniform weighting is essential if $\overline{(\ )}^L$ is to correspond to Stokes' original concept of Lagrangian mean when the latter is approximately applicable.

The definition (7.1) can be used as it stands for taking averages on a sphere, provided that the quantity ϕ being averaged is a scalar. When it is a vector or a tensor, further discussion is needed (McIntyre,1979§4).

The relation (7.1) is also very suggestive of how to define a "modified GLM" independent of initial conditions and therefore not subject to the temporally nonuniform behaviour noted at the end of section 4. This could be a matter of some importance for using the theory to describe the long-term behaviour of pollutants and planetary-wave activity in the stratosphere. The idea arose during conversations with T.Dunkerton. Consider a hypothetical motion in which $\underset{\sim}{X}$ and Q are zero, for all time, so that the motion is conservative (and has been ever since the initial state of no disturbance). Then since entropy S and potential vorticity P are both constant following this hypothetical motion, we could have defined our thin material tube C to be a tube bounded by surfaces of constant S and S + ΔS , and P and P + ΔP . (The tube would then have a cross-section approximately in the form of a parallelogram, as suggested in Fig.5c. — except in the singular case where S and P surfaces coincide.) Now in the real stratosphere S and P are not constant following the motion, because of radiative-photochemical effects and turbulent dissipation; but we could still define a "modified GLM operator" by (6.1) with the tube C still marked out by the S and P surfaces, knowing that the modified GLM theory would have the same mathematical structure as the theory described in AM, apart from the dissipative effects. This effectively provides a continuous re-initialization which eliminates the temporal nonuniformity and expresses the distinction between wave transience and dissipation in an intuitively more satisfactory way.

On the other hand the topology of such "SP tubes" could become complicated during strong disturbances. In studies of individual disturbed episodes it may prove better to use the SP tubes to initialise the (unmodified) GLM theory, during a less disturbed state preceding the episode in question (Dunkerton, 1979; McIntyre, 1979). Whatever procedure is adopted, it will necessarily have to cope in one way or another with the fact that nonuniform behaviour in time is a basic feature, in practice, of a description of real fluid motion using Lagrangian ideas in any form. The detailed study of this temporal nonuniformity, and the associated dispersal of fluid particles, is sure to play a key role in understanding exactly how the various 'nonacceleration' and 'nontransport' constraints are broken by real, large-amplitude stratospheric motions.

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