TRANSIENT AND ASYMPTOTIC ERROR GROWTH IN TIME DEPENDENT FLOWS

B F Farrell
Harvard University
Cambridge, USA

P J Ioannou
University of Athens
Athens, Greece

Abstract

Stability of time dependent flows is examined extending methods recently developed for analyzing stability of time independent flows. These methods approach the stability problem by analyzing the non-normality of the underlying dynamical operator. For both autonomous and non-autonomous operators this approach leads to identification of a complete set of optimal perturbations ordered according to extent of growth over a chosen time interval as measured in a chosen norm. The long time asymptotic structure in the case of an autonomous operator is the norm independent least stable normal mode while in the case of the non-autonomous operator it is the first Lyapunov vector which is also norm independent and grows exponentially in the mean at the rate of the first Lyapunov exponent. While structure, growth rates and energetics of the normal mode and therefore the asymptotic stability properties of autonomous systems are easily accessible through eigenanalysis of the associated dynamical operator, analogous information for the Lyapunov vector is less readily obtained. In this work the stability of time dependent deterministic and stochastic dynamical operators is examined in order to obtain a better understanding of the dynamics of asymptotic instability in time dependent systems. It is found that the physical mechanism producing asymptotic error growth in time dependent systems can be traced to the generic non-normality of the non-autonomous operator. Implications for the Lyapunov exponent magnitude and associated vector structure in tangent linear equations for forecast error growth are discussed

1 INTRODUCTION

Linear stability theory addresses a set of phenomena in the dynamics of the atmosphere which include the origin, energetics, structure and growth to finite amplitude of perturbations ultimately responsible for such macroscopic phenomena as cyclones and upper level waves. In addition these same methods are used to approach the conceptually distinct problem of predictability and error growth in which the focus is on the rate of divergence of initially nearby trajectories in the configuration space of the forecast model. While the operator in both the stability and the predictability problems is in general time dependent, traditional stability analysis confined attention to the $t \to \infty$ asymptotic of the least stable mode in autonomous dynamical operators. However, recent generalizations of stability theory have addressed the more physically relevant finite time stability problem for both autonomous and non-autonomous systems (*Farrell and Ioannou*, 1996a, 1996b). In the case of an autonomous operator, asymptotic error growth or growth of an initially arbitrarily small perturbation occurs at the rate of the least stable eigenvalue of the linear dynamical operator and takes the form of the associated normal mode. The analogous structure and growth rate in the case of a non-autonomous operator are given by the time dependent first Lyapunov vector for the structure and in the mean by the first Lyapunov exponent for the growth (*Oseledec*, 1968). While asymptotic structure and growth are readily obtained for autonomous systems through eigenanalysis of the dynamical operator and a variety of theorems typically involving the vorticity distribution of the background state are known to constrain asymptotic stability of autonomous systems (*Rayleigh*, 1800; *Fjortoft*, 1950) these rather strict results are not available for non-autonomous systems in which constraints on growth take the form of more general energy bounds which are less restrictive. Moreover, growth rates and structures of Lyapunov vectors for realistic physical systems are not commonly available, presumably because of the nongeneric nature of time dependent systems; consequently, intuition concerning such issues as the variability with time of Lyapunov vectors and their energetics lacks example. Given that all physical systems are to a greater or lesser extent time dependent and that the atmosphere in particular is highly time dependent it is of more than strictly theoretical interest to understand better asymptotic stability in such systems. Indeed, forecast accuracy is ultimately limited by asymptotic error growth rate and the time at which this limit is approached starting from a sufficiently small perturbation as well as the universal structure taken by the disturbance in this limit have important practical implications.

2 STABILITY THEORY FOR NON-AUTONOMOUS DYNAMICAL SYSTEMS

The linear time dependent dynamical system

$$\frac{du}{dt} = \mathbf{A(t)}\, u \,, \tag{1}$$

has solution :

$$u(t) = \mathbf{\Phi}_{[t,t_0]}\, u(t_0) \,. \tag{2}$$

The propagator $\mathbf{\Phi}_{[t,t_0]}$ maps the state of the system at time t_0 to its state at time t. It is given by the following time ordered exponential:

$$\mathbf{\Phi}_{[t,t_0]} = \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(s)ds + \int_{t_0}^{t} \mathbf{A}(r)dr \int_{t_0}^{r} \mathbf{A}(s)ds + \cdots \,, \tag{3}$$

which is equivalent in the limit to the ordered product of infinitesimal propagators :

$$\Phi_{[t,t_0]} = \lim_{\delta t \to 0} \prod_{j=1}^{n} e^{A(t_j)\delta t} , \tag{4}$$

for t_j lying in the mesh $t_0 + (j-1)\delta t < t_j < t_0 + j\delta t$ and $t = t_0 + n\delta t$. The propagator obeys the semigroup property: $\Phi_{[t,s]} \, \Phi_{[s,t_0]} = \Phi_{[t+s,t_0]}$ and solves

$$\frac{d\Phi_{[t,t_0]}}{dt} = \mathbf{A}(t) \, \Phi_{[t,t_0]} \quad \text{with} \quad \Phi_{[t_0,t_0]} = \mathbf{I} \, . \tag{5}$$

Analysis of the stability properties of this dynamical system differs from analysis of the stability of autonomous operators in that system (5) has asymptotic behavior that can not be determined by examining the behavior of the temporal eigenmodes of $\mathbf{A}(t)$ which are not defined for general time dependence. However, the optimal perturbations retain their meaning and provide the required description of the stability of the system for all time. In fact it remains possible to uniquely define an asymptotic exponential rate of growth or decay for time dependent operators which is the first Lyapunov exponent (*Lyapunov*, 1907) defined as

$$\lambda = \lim_{t \to \infty} \sup \frac{\ln(\| \Phi(t) \|)}{t} . \tag{6}$$

The first Lyapunov exponent assumes the role played by the most rapidly growing mode in autonomous systems and reduces to the most rapidly growing mode in the limit that the system becomes independent of time. Any dynamical system with $\lambda > 0$ is asymptotically unstable.

If we assume that the time dependent operator is comprised of a sum of a time independent mean operator and a stochastic operator then an extension of the Lyapunov results of *Oseledec* (1968) establishes the existence of unique Lyapunov exponents almost surely in probability. All the properties established for deterministic operators carry over to the stochastic case with the exception that the first Lyapunov vector and its optimal excitation depend on the specific time realization while the associated first Lyapunov exponent is independent of the particular realization (*Arnold and Kliemann*, 1983).

3 PARAMETRIC INSTABILITY IS A CONSEQUENCE OF THE NON-NORMALITY OF THE OPERATOR

We obtain a bound on the Lyapunov exponent by considering the evolution of $r(t) = \| u(t) \|^2$ which is easily seen to obey :

$$\frac{dr}{dt} = u^\dagger(\mathbf{A}(t) + \mathbf{A}^\dagger(t))u \tag{9}$$

implying

$$2\int_0^t \lambda_{\min}(s)ds \leq \log\left(\frac{r(t)}{r(0)}\right) \leq 2\int_0^t \lambda_{\max}(s)ds , \tag{10}$$

where $\lambda_{\max}$ and $\lambda_{\min}$ are the maximum and minimum eigenvalues of the Hermitian operator $(\mathbf{A}(t) + \mathbf{A}^\dagger(t))/2$. This leads to the bound for the Lyapunov exponent:

$$\limsup_{t\to\infty} \frac{\int_0^t \lambda_{\min}(s)ds}{t} \leq \lambda \leq \limsup_{t\to\infty} \frac{\int_0^t \lambda_{\max}(s)ds}{t} . \tag{11}$$

For autonomous dynamical systems the right inequality leads to the energy bound of *Joseph* (1976) based on the numerical range λ_{max} which is typically indicative of the optimal growth only for very small times after which decay ensues if the operator is asymptotically stable. It is possible for time dependent operators to forestall this decay and sustain some portion of the instantaneous growth predicted by the numerical range leading in the asymptotic limit to instability. This process is referred to as parametric instability and is exemplified by the destabilization of the harmonic oscillator by sinusoidal perturbation of its restoring force; an instability familiarly associated with the Mathieu equation. Despite its at most neutral stability at each time instant the time dependent operator associated with the perturbed harmonic oscillator is unstable for specific intervals in frequency of the restoring force perturbation. It is immediate from (11) that non-normality of the evolution operator is a necessary condition for parametric asymptotic instability to occur in an operator which is stable at each instant of time.

As an example of the role of non-normality in parametric instability consider the harmonic oscillator with time dependent acceleration proportional to displacement:

$$\frac{d}{dt}\begin{pmatrix} x \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2(t) & -2\gamma \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} . \tag{12}$$

The operator in (12) has commutator:

$$\mathbf{A}\,\mathbf{A}^\dagger - \mathbf{A}^\dagger\,\mathbf{A} = \begin{pmatrix} 1-\omega^4(t) & -2\gamma(1+\omega^2(t)) \\ 2\gamma(1+\omega^2(t)) & 1-\omega^4(t) \end{pmatrix} , \tag{13}$$

indicating that $\mathbf{A}$ is non-normal except for $\gamma = 0$ and ω constant and equal to unity. If the frequency is constant and $\gamma = 0$ we are at liberty to rescale time by $1/\omega$ to make $\omega = 1$ and render the operator formally normal in the L_2 norm. This scaling is equivalent to the coördinate transformation : $\tilde{x} = x$, $\tilde{v} = v/\omega$. But such a transformation with constant ω can not succeed in making the operator uniformly normal when the operator is time dependent leaving open the possibility of positive Lyapunov exponents according to (11). It can be shown that this destabilization persists even when $\gamma > 0$ so that the operator is stable at each time instant.

This synergism of non-normality and time dependence leading to asymptotic instability can be succinctly demonstrated by considering a discontinuous change in ω between ω_1 and ω_2 every T units of time. With $\gamma = 0$ the propagator after a full period takes the form:

$$\mathbf{\Phi}_{2T} = \prod_{i=1,2} \left[\mathbf{I}\,\cos(\omega_i T) + \mathbf{A}_\mathrm{i}\,\frac{\sin(\omega_i T)}{\omega_i}\right] , \tag{14}$$

with $\mathbf{I}$ the identity and

$$\mathbf{A}_\mathrm{i} = \begin{pmatrix} 0 & 1 \\ -\omega_i^2 & 0 \end{pmatrix} . \tag{15}$$

The first Lyapunov exponent can be readily calculated from

$$\lambda = \lim_{n\to\infty} \frac{\log(\|\,\mathbf{\Phi}_{2T}^n\,\|)}{2nT} , \tag{15}$$

where n is the number of periods. Unless the switchover time T is near an integer multiple of one-half the natural period of one of the oscillations it can be verified that positive Lyapunov exponents are obtained (Fig. 1). The reason for the instability is that transient growth instigated at the starting time is continued with further growth when the switch to the second operator takes place forestalling the decay which would occur in the autonomous case.

It is natural to inquire whether this support of parametric instability by non-normal systems with periodic parameter modulation leads also to asymptotic instability for parameter modulations of more general form. One limit of parametric modulation is stochastic modulation of the system's parameters and it is remarkable that under the broadest assumptions stochastic modulation of the restoring force leads to asymptotic instability (*Arnold et al.*, 1986; *Colonius and Kliemann*, 1993). In the sequel we inquire whether this generic destabilization carries over to highly non-normal operators such as those governing evolution of perturbations to the large scale atmospheric flow.

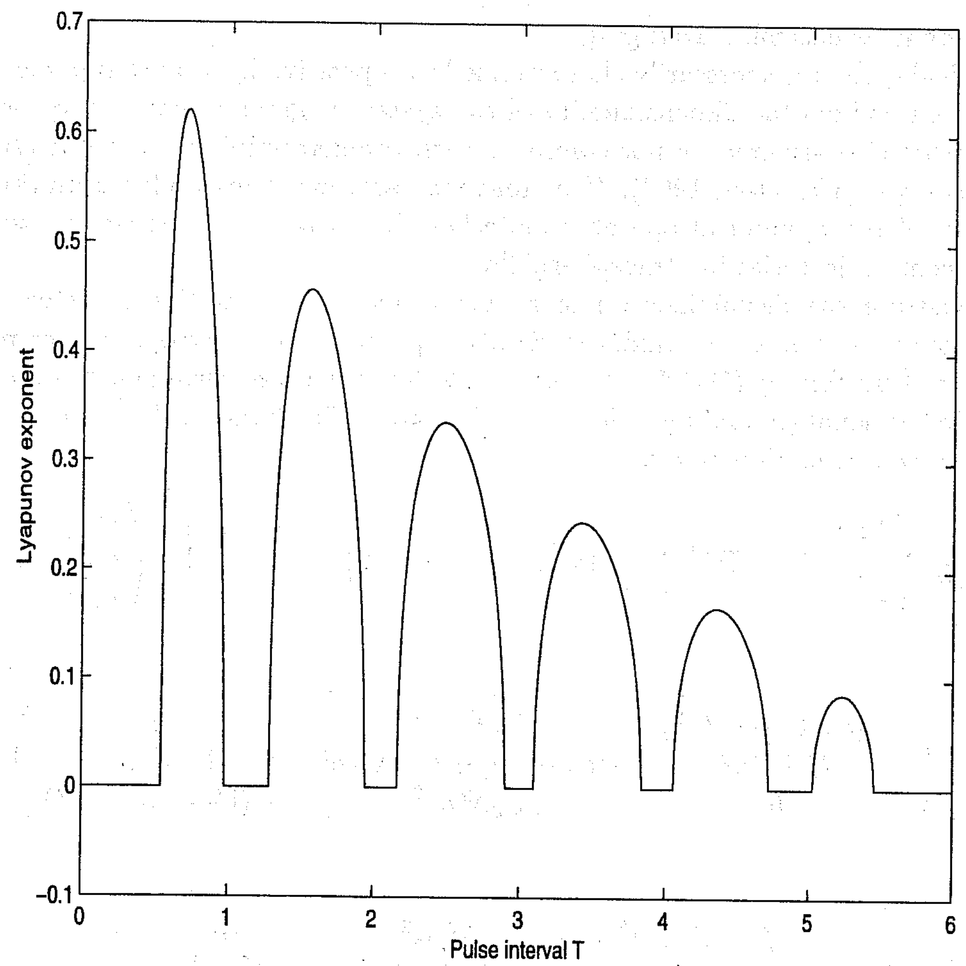

Figure 1: Lyapunov exponent for the switched oscillator example as a function of the period of the switch T nondimensionalized by π/ω_2 for $\omega_1 = 0.5$ and $\omega_2 = 3$. Although instantaneously neutral at all times, the oscillator is destabilized except in the neighborhood of integer values of the switching period.

4 STABILITY OF TIME DEPENDENT ATMOSPHERIC FLOWS

The tangent linear system governing the evolution of small initial perturbations to a solution of the nonlinear equations of motion can be cast as a non-autonomous linear system of form (1). We can decompose this non-autonomous operator as $\mathbf{A}(t) = \overline{\mathbf{A}} + \mathbf{A}'(t)$ where $\overline{\mathbf{A}}$ is the time mean operator and $\mathbf{A}'(t)$ is the operator arising from fluctuations which will be taken to be stochastic. In the case of the atmosphere $\overline{\mathbf{A}}$ is the mean dynamical operator which is asymptotically stable. We model the tangent linear system as:

$$\frac{du}{dt} = \overline{\mathbf{A}}\, u + \epsilon \sum_i \xi_i(t) \mathbf{B}_i\, u \tag{16}$$

where ϵ is the r.m.s. magnitude of the fluctuations, $\mathbf{B}_i$ are time independent noise matrices, and the noise processes $\xi_i(t)$ are identically distributed processes with zero mean and with covariances :

$$< \xi_i(t)\ \xi_j(s) > = \delta_{ij}\ \delta(t-s)\ , \tag{17}$$

where $< . >$ denotes ensemble averaging.

A dynamical system is necessarily characterized by a positive Lyapunov exponent for strong enough noise ϵ provided the dimensionality of the system is greater than 2 and the noise matrices are neither skew-symmetric nor commute with the deterministic operator (*Has'minskii*, 1980; *Colonius and Kliemann*, 1993). This universal stochastic instability resulting from the non-normality of the dynamical operator underlies the asymptotic increase in separation of initially adjacent trajectories in atmospheric flows.

To demonstrate the destabilization of an atmospheric tangent linear system consider a three layer approximation of the midlatitude atmosphere. The geostrophic streamfunction is assumed to be of the form $\psi_i(t)\ e^{ikx\ +ily}$ with i=1, 2, 3 in the three vertical layers and harmonic dependence in the zonal (x) and meridional (y) direction. The dynamical equation for the three components of the streamfunction is:

$$\frac{d}{dt}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix} = \mathbf{P}^{-1}\left[-ik\left(\mathbf{A} + \frac{\beta}{\alpha^2}\,\mathbf{I}\right) - r\,\mathbf{I}\right]\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}\ ; \tag{27}$$

with

$$\mathbf{A} = \begin{pmatrix} -(U_1+U_2\lambda^2/\alpha^2) & U_1\lambda^2/\alpha^2 & 0\\ U_2\lambda^2/\alpha^2 & -(U_2+(U_1+U_3)\lambda^2/\alpha^2) & U_2\lambda^2/\alpha^2\\ 0 & U_2\lambda^2/\alpha^2 & -(U_3+U_2\lambda^2/\alpha^2)\end{pmatrix}, \tag{28}$$

$$\mathbf{P} = \begin{pmatrix} -(1+\lambda^2/\alpha^2) & \lambda^2/\alpha^2 & 0\\ \lambda^2/\alpha^2 & -(1+\lambda^2/\alpha^2) & \lambda^2/\alpha^2\\ 0 & \lambda^2/\alpha^2 & -(1+\lambda^2/\alpha^2)\end{pmatrix}. \tag{29}$$

and $\mathbf{I}$ the identity matrix.

The top layer is indexed first and each layer velocity is considered to be of the form $U_i = \overline{U_i} + \epsilon\ \xi_i(t)$, with ξ_i being an independent white noise process as in (17). The mean velocities (m/s) have been expressed in terms of the mean shear over each layer, ΔU, as $\overline{U_i} = 10 + (3-i)\Delta U$ ($i = 1,2,3$). The total horizontal wavenumber is $\alpha = (k^2 + l^2)^{1/2}$; the meridional wavelength $l = \pi/Y_c$ corresponds to the gravest mode in a channel of meridional extent $Y_c = 4000\ km$. The Rossby deformation wavenumber is $\lambda = f_0/(\sigma^{1/2}\delta p)$ with $f_0 = 10^{-4}\ s^{-1}$ and $\beta = 1.65 \times 10^{-11}\ m^{-1}s^{-1}$, the midlatitude value of Coriolis parameter and its northward derivative respectively, and $\sigma = 2 \times 10^{-6}\ Pa^{-2}s^{-2}m^{-2}$ the stratification parameter typical of the troposphere. Equally spaced pressure surfaces have been taken with $\delta p = 10^5/3\ Pa$. The coefficient of potential vorticity damping is denoted by r. Equations (27), (28) are presented in non-dimensional form. Time has been non-dimensionalized by $T_d = 1$ day; horizontal lengths by L_x the perimeter of the latitude circle at 45°; and velocities by L_x/T_d.

The first Lyapunov exponent as a function of shear ΔU and r.m.s. noise variance ϵ for global wavenumber 11 and non-dimensional potential vorticity damping $r = 0.2$ corresponding to an e-folding of 5 days is shown in Fig. 2. Note the destabilization of this system as noise increases. The threshold noise required to destabilize the system is seen to gradually decrease as the non-normality of the mean operator, indicated by the shear, increases. The typical midlatitude shear corresponds in this three layer model to $\Delta U = 10\ m/s$. This simplified model of the midlatitude atmosphere suggests existence of a positive Lyapunov exponent for r.m.s. temporal fluctuations of the order of 10%, while for 30% fluctuations a Lyapunov exponent of the order of $1/5\ day^{-1}$ is expected.

Although useful for constructing simple examples, white noise forcing is unrealistic for modeling physical systems in that maintaining finite variance on passing to the limit of delta correllation of the noise requires unbounded amplitude of the forcing. More realistic modeling requires using temporal variation with finite correlation time in the operator which permits the ampitude of the forcing to remain bounded.

5 Discussion

Two distinct problems are addressed by analysis of the linear stability of time dependent operators: the growth of errors and the growth of perturbations. Calculation of error growth usually involves the time dependent system of tangent linear equations in which the linearization has been performed about a known time dependent trajectory and the perturbation is regarded as a small error in specifying the initial conditions. The result of the calculation is the difference between the perturbed and the unperturbed trajectories and it is valid until nonlinear effects become important. If a positive Lyapunov exponent exists then an arbitrarily small perturbation to initial conditions eventually produce an order one change in the trajectory and after a period of adjustment this change assumes the form of the time dependent first Lyapunov vector. In this case the asymptotic stability calculation is interpreted as constraining the predictability of the system.

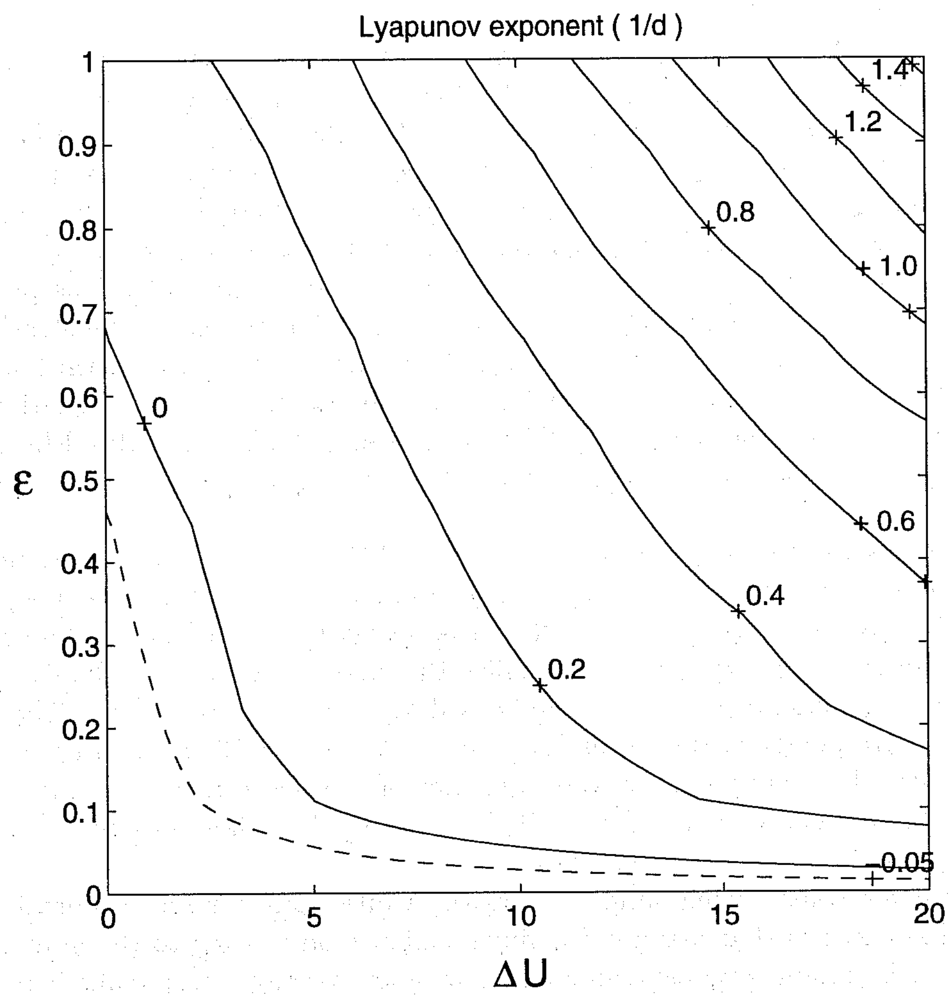

Figure 2: The Lyapunov exponent for the baroclinic three-layer model as a function of shear and strength of the multiplicative noise forcing. The global zonal wavenumber at 45^o of latitude is 11, the meridional wavenumber is $l = \pi/4$, and dissipation corresponding to $r = 0.2$ has been included. A finite magnitude of both shear and parametric forcing is required to produce a positive Lyapunov exponent. The Lyapunov exponent increases with both the shear and the magnitude of multiplicative noise forcing.

The growth of perturbations problem by contrast envisions development to finite amplitude of a disturbance on a time dependent flow such as the atmospheric jet and the result is interpreted as a generalization of the transient growth to finite amplitude of perturbations on stationary flows. An example of this type of problem would be an instability calculation studied as a model for cyclogenesis. Such instability problems have most often been examined assuming that the underlying operator to be autonomous. This assumption is seldom critically evaluated and it would be of interest, for example, to find that the mid-latitude jet while stable if its time mean is analyzed were found to be unstable if realistic vacillation were included in its specification. It could then be that the stable mean jet spontaneously gives rise to perturbations in the form of the Lyapunov vector. The nature of these perturbations and their relationship to cyclogenesis and the maintenance of long waves would then be of great interest to understanding the general circulation of the atmosphere.

Given that all physical problems are to a greater or lesser extent time dependent it follows that the stability of realistic flows must take account of time dependence in some way. For short enough time intervals the time dependence of forecast model tangent linear trajectories may be ignored without undue effect on the calculation of forecast error growth; moreover, the dominant energetics of the mid-latitude atmosphere appear to be associated with growth on short enough time scales that the statistics of the atmosphere can be accurately modeled using autonomous time mean operators (*Farrell and Ioannou*, 1993). However, for longer time periods such as are associated with medium and longer range forecast and with asymptotic error growth, explicit account of time dependence needs to be taken. To this end the methods of non-normal dynamics can be straightforwardly extended from the study of growth of perturbations over finite time in autonomous systems to study the growth of perturbations over finite time in non-autonomous systems. The optimal perturbations and the structures into which these evolve over a given interval of time can be obtained by singular value decomposition of the propagator whether the propagator arises from an autonomous or a non-autonomous system. In the limit of long time the analogue of the most rapidly growing normal mode which asymptotically dominates in the autonomous system is the first Lyapunov vector which asymptotically dominates in the non-autonomous system. This asymptotic growth in the non-autonomous system can also be analyzed through the non-normal dynamics of the underlying dynamical operator given a specification of the time dependence of the system. But this raises an issue concerning the methods required in the study of non-autonomous operators. While a stationary state can be easily specified by a deterministic function, a time dependent system such as the atmosphere can only be specified in general terms through its statistical properties. We would like to turn this limitation to advantage to obtain results that transcend the particularities of a given realization of the system to reveal rather the general statistical properties of the system's time dependent stability. We have studied this problem by using stochastic perturbation of the mean operator with the perturbations chosen to model statistically the observed deviations from the mean of the atmospheric jet within the limitations of simple models. We found that asymptotic stability properties of time dependent systems can be understood in general terms through considerations employed by *Zeldovitch et al.* (1984) in his analysis of vector growth in random unitary systems; that is instability results because growth of a randomly chosen vector over a finite time interval can dominate over decay even when the development of a randomly chosen vector over the numerical range results in decay. For a given temporal correlation interval this mechanism can be modeled by extension of Floquet analysis from a periodic to an aperiodic model system (*Farrell and Ioannou*, 1996b).

Using this theory and given statistical properties of the time dependence of the operator, the asymptotic stability properties and the characteristics of the associated Lyapunov vector can be obtained. This allows us to evaluate circumstances under which destabilization of the system due to time dependence is likely and to understand the resulting growing structures. Some further results using this approach follow (*Farrell and Ioannou*, 1998). For the example of the barotropic jet we found that increasing the amplitude of jet vacillation is destabilizing provided the vacillation does not have the same functional form as the mean flow. Increasing the spatially uniform value of constant β is stabilizing although no generalization of the Rayleigh or Fjortoft theorems exist: examples demonstrate that it is possible for a time dependent system which satisfies necessary conditions for stability at each instant of time to be unstable. Sufficient spatial and temporal variation of effective β was found to be destabilizing and this was identified as the mechanism by which variation of jet structure with time produces instability. In the case of a stable time mean operator, asymptotic stability is retained when a time dependent perturbation of bounded amplitude is added provided the temporal correlation of the time dependence of the jet vacillation is either sufficiently long or sufficiently short. The optimal correlation time for instability at an intermediate value of temporal correlation.

General considerations of destabilization of stable mean operators by time dependence show that the Lyapunov vector can not be simply the least stable mode of the mean operator which has been destabilized by time dependence; rather, a mixture of modes in the non-orthogonal subspace of the operator must take part in producing the Lyapunov vector and as a result the asymptotic instability possesses a character distinct from the individual modes.

It is not necessary for a turbulent flow to have a positive Lyapunov exponent. Consider a forced but highly damped flow; the forcing can produce an arbitrarily complex flow field which is completely determined by the, in principle, known forcing. Meanwhile the damping could in principle be sufficiently great that no positive Lyapunov exponent exists. The atmosphere in summer may be such a forced system. While this would mean that an arbitrarily small perturbation would not result in a completely different atmospheric state after sufficient time has passed it would not mean that the atmospheric state would be predictable given that the forcing; arising e.g. from latent heat release, is unknown.

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