SEMI-LAGRANGIAN METHODS

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Summary: The semi-Lagrangian method is reviewed for a hierarchy of applications (passive advection, forced advection and coupled sets of equations) of increasing complexity in one, two and three dimensions.

1. INTRODUCTION

Accurate and timely forecasts of weather elements are of great importance to both public safety and to the world economy. The accuracy of computer-generated weather forecasts depends among other things on model resolution. Increased resolution, given the real-time constraints, can only be achieved by combining the most efficient numerical methods on the most powerful computers with the most appropriate programming techniques.

A longstanding problem in the integration of Numerical Weather Prediction (NWP) models is that the maximum permissible timestep has been governed by considerations of stability rather than accuracy. For the integration to be stable, the timestep has to be so small that the time truncation error is much smaller than the spatial truncation error, and it is therefore necessary to perform many more timesteps than would otherwise be the case. The choice of time integration scheme is therefore of crucial importance when designing an efficient atmospheric model. Early NWP models used an explicit leapfrog scheme, whose timestep is limited by the propagation speed of gravitational oscillations. By treating the linear terms responsible for these oscillations in an implicit manner, it is possible to lengthen the timestep by several factors, at little additional cost and without degrading the accuracy of the solution (e.g. Robert, 1969; Robert et al., 1972). Such a scheme is termed *semi-implicit*. Nevertheless, the maximum stable timestep still remains much smaller than seems necessary from considerations of accuracy alone (Robert, 1981).

Discretisation schemes based on a semi-Lagrangian treatment of advection have elicited considerable interest in the past decade for the efficient integration of atmospheric models, since they offer the promise of allowing larger timesteps (with no loss of accuracy) than Eulerian-based advection schemes (whose timestep length is overly limited by considerations of stability). To achieve this end it is essential to associate a semi-Lagrangian treatment of advection with a sufficiently-stable treatment of the terms responsible for the propagation of gravitational oscillations. By associating a *semi-Lagrangian* treatment of advection with a *semi-implicit* treatment of gravitational oscillations, Robert (1981, 1982) demonstrated a further significant increase in the maximum stable timestep, at the cost of performing some upstream interpolations. This idea was demonstrated in the context of a *three*-time-level shallow-water finite-difference model in Cartesian geometry, and resulted in the time truncation errors finally being of the same order as the spatial ones.

Since Robert's seminal papers, the semi-Lagrangian methodology for advection-dominated fluid flow problems has been extended in several important ways and is now widely used. The purpose of this paper is to summarise the fundamentals of semi-Lagrangian advection (Section 2), to describe its application to coupled sets of equations (Section 3), and to summarise some of the early extensions of the method (Section 4).

2. SEMI-LAGRANGIAN ADVECTION

In an *Eulerian* advection scheme an observer watches the world evolve around him at a fixed geographical point. Such schemes work well on regular meshes (facilitating vectorisation and parallelisation of the resulting code), but often lead to overly-restrictive timesteps due to considerations of computational stability. In a *Lagrangian* advection scheme an observer watches the world evolve around him as he travels with a fluid particle. Such schemes can often use much larger timesteps than Eulerian ones, but have the disadvantage that an initially regularly-spaced set of particles will generally evolve to a highly-irregularly-spaced set at later times, and important features of the flow may consequently not be well represented. The idea behind *semi-Lagrangian* advection schemes is to try to get the best of both worlds: the regular resolution of Eulerian schemes and the enhanced stability of Lagrangian ones. This is achieved by using a different set of particles at each timestep, the set of particles being chosen such that they arrive exactly at the points of a regular Cartesian mesh at the end of the timestep. This idea evolved from pioneering work of the fifties and sixties (Staniforth and Côté, 1991).

2.1 **Passive advection in 1-d**

Consider the 1-d advection equation

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{dx}{dt}\frac{\partial F}{\partial x} = 0, \tag{1}$$

where

$$\frac{dx}{dt} = U(x,t), \tag{2}$$

and $U(x,t)$ is a given function. Eq. (1) states that the scalar F is constant along a fluid path (or trajectory or characteristic).

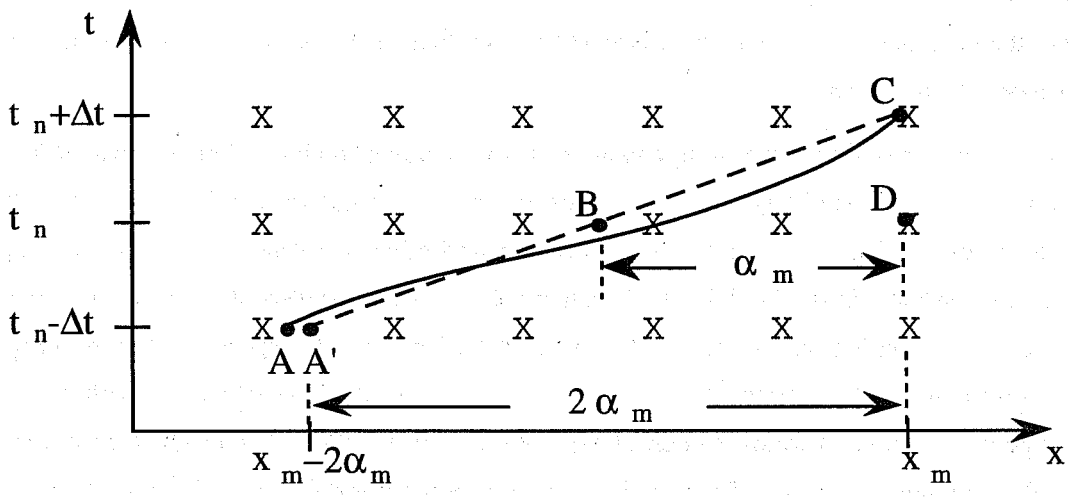

Fig. 1: Schematic for 3-time-level advection. Actual (solid curve) and approximated (dashed line) trajectories that arrive at meshpoint x_m at time $t_n + \Delta t$. Here α_m is the distance the particle is displaced in x in time Δt.

In Fig. 1, the *exact* trajectory in the (*x-t*) plane of the fluid particle that arrives at meshpoint x_m at time $t_n + \Delta t$ is denoted by the solid curve AC, and an *approximate* straight-line trajectory by the dashed line A'C. Assume that $F(x,t)$ is known at all meshpoints x_m at times $t_n - \Delta t$ and t_n, and that we wish to obtain values at the

same meshpoints at time $t_n + \Delta t$. The essence of semi-Lagrangian advection is to approximately integrate (1) along the approximated fluid trajectory A'C. Thus

$$\frac{F(x_m, t_n + \Delta t) - F(x_m - 2\alpha_m, t_n - \Delta t)}{2\Delta t} = 0, \tag{3}$$

where α_m is the distance BD the particle travels in x in time Δt, when following the approximated space-time trajectory A'C. Thus if we know α_m, then the value of F at the arrival point x_m at time $t_n + \Delta t$ is just its value at the upstream point $x_m - 2\alpha_m$ at time $t_n - \Delta t$. However α_m is not as yet determined : even if it were, F is only known at meshpoints, and generally it still remains to evaluate it somewhere between meshpoints.

To determine α_m, note that U evaluated at the point B of Fig. 1 is just the inverse of the slope of the straight line A'C, and this leads to the following $O(\Delta t^2)$ approximation to (2) (Robert, 1981):

$$\alpha_m = \Delta t U(x_m - \alpha_m, t_n). \tag{4}$$

Eq. (4) may be iteratively solved for the displacement α_m, for example by

$$\alpha_m^{(k+1)} = \Delta t U(x_m - \alpha_m^{(k)}, t_n), \tag{5}$$

with some initial guess for $\alpha_m^{(0)}$, provided U can be evaluated between meshpoints. To evaluate F and U between meshpoints, spatial interpolation is used. The semi-Lagrangian algorithm for passive advection in 1-d in summary is thus:

(i) Solve (5) iteratively for the displacements α_m for all meshpoints x_m, using some initial guess (usually its value at the previous timestep), and an interpolation formula.

(ii) Evaluate F at upstream points $x_m - 2\alpha_m$ at time $t_n - \Delta t$ using an interpolation formula.

(iii) Evaluate F at arrival points x_m at time $t_n + \Delta t$ using (3).

We defer the discussion of interpolation details to Section 2.4, and first generalise the above three-time-level algorithm to forced advection in several space dimensions (Section 2.2), and to *two* time levels (Section 2.3).

2.2 Forced advection in multi-dimensions

Consider the forced advection problem

$$\frac{dF}{dt} + G(x,t) = R(x,t), \tag{6}$$

where

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \mathbf{V}(\boldsymbol{x},t)\cdot\nabla F, \tag{7}$$

$$\frac{d\boldsymbol{x}}{dt} = \mathbf{V}(\boldsymbol{x},t), \tag{8}$$

$\boldsymbol{x}$ is the position vector (in 1-, 2- or 3-d), ∇ is the gradient operator, and G and R are forcing terms. A semi-Lagrangian approximation to (6) and (8) is then:

$$\frac{F^+ - F^-}{2\Delta t} + \frac{1}{2}\left[G^+ + G^-\right] = R^0, \tag{9}$$

$$\alpha = \Delta t \mathbf{V}(x - \alpha, t), \tag{10}$$

where the superscripts "+", "0" and "-" respectively denote evaluation at the arrival point $(x, t+\Delta t)$, the midpoint of the trajectory $(x - \alpha, t)$ and the departure point $(x - 2\alpha, t-\Delta t)$. Here, x is now an arbitrary point of a regular (1-, 2- or 3-d) mesh.

The above is a centered $O(\Delta t^2)$ approximation to (6) and (8), where G is evaluated as the time average of its values at the endpoints of the trajectory, and R is evaluated at the midpoint of the trajectory. The trajectories are calculated by iteratively solving (10) for the *vector* displacements α in a manner analogous to the 1-d case for passive advection [eq. (5)]. If G is known (we assume that R is known since it involves evaluation at time t), then the algorithm proceeds in an analogous manner to the 1-d passive advection one and is thus:

(i) Solve (10) iteratively for the *vector* displacements α for all meshpoints x, using some initial guess (usually its value at the previous timestep), and an interpolation formula.

(ii) Evaluate $F - \Delta t G$ at upstream points $x - 2\alpha$ at time $t - \Delta t$ using an interpolation formula. Evaluate $2\Delta t R$ at the midpoints $x - \alpha$ of the trajectories at time t using an interpolation formula.

(iii) Evaluate F at arrival points x at time $t + \Delta t$ using

$$\begin{aligned} F(x, t+\Delta t) &= (F - \Delta t G)\big|_{(x-2\alpha, t-\Delta t)} + 2\Delta t R\big|_{(x-\alpha, t)} - \Delta t G\big|_{(x, t+\Delta t)} \\ &= (F - \Delta t G)^- + 2\Delta t R^0 - \Delta t G^+. \end{aligned} \tag{9'}$$

If G is not known at time $t + \Delta t$ (for instance if it involves another dependent variable in a set of coupled equations), then this leads to a coupling to other equations (more on this in Section 3).

2.3 Two-time-level advection schemes

Present semi-Lagrangian schemes are usually based on discretisation over either *two* or *three* time levels. Thus far we have restricted our attention to three-time-level schemes. The principal advantage of *two-time-level* schemes over three-time-level ones is that they are potentially *twice* as fast. This is because three-time-level schemes require timesteps half the size of two-time-level ones for the same level of time truncation error (Temperton and Staniforth , 1987). It is however important to maintain second-order accuracy in time in order to reap the full benefits of a two-time-level scheme (since enhanced stability with large timesteps is of no benefit if it is achieved at the expense of diminished accuracy). Early two-time-level schemes for NWP models unfortunately suffered from this deficiency. The crucial issue is how to efficiently determine the trajectories to at least second-order accuracy in time (Staniforth and Pudykiewicz, 1985; McDonald, 1987).

This problem arises in the context of self-advection of momentum. To see this we reexamine the algorithm of Section 2.1 for 1-d advection. Provided U is known at time t_n, *independently of F at the same time*, then it is possible to evaluate the trajectory, and then leapfrog the value of F from time $t_n - \Delta t$ to $t_n + \Delta t$ without knowing any value of F at time t_n. Proceeding in this way, $F(t_n + 3\Delta t)$ is then obtained using values of $F(t_n + \Delta t)$ and $U(t_n + 2\Delta t)$. Thus we have two decoupled independent integrations, one using values of F at

even timesteps and U at *odd* timesteps, the other using values of F at *odd* timesteps and U at *even* timesteps. Either of these two independent solutions is sufficient, thus halving the computational cost, and we obtain a two-time-level scheme (for the advected quantity F) by merely relabelling time levels $t_n - \Delta t$, t_n and $t_n + \Delta t$ respectively as t_n, $t_n + \Delta t/2$ and $t_n + \Delta t$ (see Fig. 2). Note that values of U (assumed known) only appear at time level $t_n + \Delta t/2$, and they are *solely* used to estimate the trajectories. This is the essence of the 2-d advection-diffusion algorithm described and analysed in Pudykiewicz and Staniforth (1984).

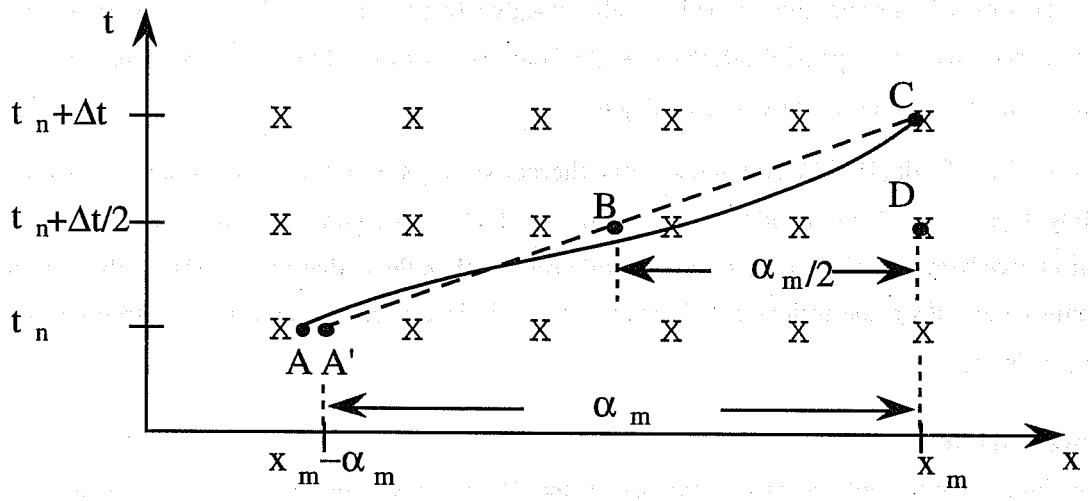

Fig. 2: Schematic for 2-time-level advection. Actual (solid curve) and approximated (dashed line) trajectories that arrive at meshpoint x_m at time $t_n + \Delta t$. Here α_m is the distance the particle is displaced in x in time Δt.

Returning to the problem of self-advection of momentum, the above argument breaks down in the special case where $F = U$ in (1) or $F = \mathbf{V}$ in (6), i.e. when the transported quantity U or $\mathbf{V}$ is advected by itself, as is the case for the momentum equations of fluid-dynamic problems in general, and NWP models in particular. This problem was addressed simultaneously and independently by Temperton and Staniforth (1987) and McDonald and Bates (1987), opening the way towards stable *and* accurate two-time-level schemes. The key idea here is to time-extrapolate the winds [with an $O(\Delta t^2)$-accurate extrapolator] to time-level $t + \Delta t/2$ using the known winds at time levels t and $t - \Delta t$ these winds are then used to obtain sufficiently-accurate [$O(\Delta t^2)$] estimates of the trajectories, which in turn are used to advance the dependent variables from time level t to $t + \Delta t$. Thus the two-time-level algorithm to solve (6)-(8), analogous to the three-time-level one given by (9)-(10), is (see Fig. 2)

$$\frac{F^+ - F^0}{\Delta t} + \frac{1}{2}\left[G^+ + G^0\right] = R^{1/2}, \tag{11}$$

where

$$\alpha = \Delta t \mathbf{V}^*(x - \alpha/2, t + \Delta t/2), \tag{12}$$

$$\mathbf{V}^*(x, t + \Delta t/2) = (3/2)\mathbf{V}(x,t) - (1/2)\mathbf{V}(x, t - \Delta t) + O(\Delta t^2), \tag{13}$$

the superscripts "+", "1/2" and "0" now respectively denote evaluation at the arrival point $(\boldsymbol{x}, t+\Delta t)$, the midpoint of the trajectory $(\boldsymbol{x} - \boldsymbol{\alpha}/2, t+\Delta t/2)$ and the departure point $(\boldsymbol{x} - \boldsymbol{\alpha}, t)$, and $\boldsymbol{\alpha}$ is still the distance the fluid particle is displaced in time Δt.

In the above formulation the evaluation of $R^{(1/2)}$ involves extrapolated quantities and therefore could potentially lead to instability. Temperton and Staniforth (1987) didn't find this to be a problem when some weak nonlinear metric effects were evaluated in this way in a shallow-water model integrated on a polar-stereographic projection. However it is preferable to evaluate all non-advective terms (i.e. G in the above) as time averages along the trajectory whenever possible. Subsequently Côté (1988) showed how to avoid evaluating the above-mentioned metric terms in terms of extrapolated quantities.

Temperton and Staniforth (1987) examined several alternative ways of extrapolating quantities for the purpose of estimating trajectories. They found that those methods which keep a particle on its exact trajectory for solid-body rotation give better results for the more general problem than those that do not. They also found that time extrapolating winds *along* the trajectory (their method 4) is less accurate than time extrapolating winds at meshpoints as in (13).

2.4 **Interpolation**

A priori any interpolation could be used to evaluate F and U (or $\mathbf{V}$) between meshpoints in the above algorithm. In practice the choice of interpolation formula has an important impact on the accuracy and efficiency of the method. Various polynomial interpolations have been tried including: linear; quadratic Lagrange; cubic Lagrange; cubic spline; and quintic Lagrange.

For step (ii) of the algorithm, it is found (for analysis see e.g.: Purnell, 1976; Bates and McDonald, 1982; McDonald, 1984; and Pudykiewicz and Staniforth, 1984) that cubic interpolation is a good compromise between accuracy and computational cost. While quadratic Lagrange interpolation is viable and was used in most of the early studies, cubic interpolation has subsequently been widely adopted. Cubic interpolation is 4th-order accurate and has very little damping (it is scale selective, with the damping affecting primarily the smallest scales), whereas linear interpolation (see McDonald, 1984, for discussion) has unacceptably-large damping (it is also scale selective, but has a much less sharp response). Purser and Leslie (1988) recommend using at least 4th-order (i.e. cubic) interpolation.

Improving the order of the interpolation formally increases the accuracy, but at additional cost, and the law of diminishing returns ultimately applies. However this is mitigated, particularly in 3-d, by the use of cascade interpolation. Cascade interpolation (Purser and Leslie, 1991) employs a sequence of 1-d interpolations to perform high-order interpolation in multi-dimensions, and uses two intersecting meshes, a regular Eulerian mesh and a curvilinear Lagrangian mesh. A typical tensor product interpolation in 3d, based on the classical 1-d Lagrange interpolator, requires $O(p^3)$ operations per gridpoint per field, where p is the formal accuracy of the interpolator. By successively interpolating the data between the Eulerian and Lagrangian meshes, this can be reduced to only $O(p)$ operations at the cost of determining a set of mesh-intersection points. As originally

proposed this overhead was quite expensive, but recently Nair et al. (1999) have shown how to significantly reduce it.

For step (i), the order of the interpolation is much less important (McDonald, 1987). It is found in practice (e.g. Staniforth and Pudykiewicz, 1985; Bates et al., 1990) that it is sufficient to use linear interpolation for the computation of the displacements, when using cubic interpolation for F, which is very economical. It is also found that there is usually no advantage in using more than two iterations for solving the displacement equation [step (i)]. McDonald (1987) has shown theoretically that it is not necessary to use the same order of interpolation for each iteration. For example, it is more economical and no less accurate to perform the first iteration using linear interpolation and the second using quadratic, than to use quadratic interpolation for both.

Pudykiewicz et al. (1985) have shown that a sufficient condition for convergence of the iterative solution of step (i) is that Δt be smaller than the reciprocal of the maximum absolute value of the wind shear in any coordinate direction. Thus

$$\Delta t < \left[\max \left(|u_x|, |u_y|, |v_x|, |v_y| \right) \right]^{-1},$$

for 2-d flow, where u and v are the two wind components. This means that the timestep of semi-Lagrangian schemes is not only limited by accuracy considerations (i.e. temporal discretisation errors) but also by properties of the flow (i.e. wind shear). However for many advection-dominated flows this is not a problem.

Although most authors have adopted polynomial schemes for the interpolatory steps of semi-Lagrangian schemes, other interpolators are also possible (e.g. Williamson and Rasch, 1989; Nair et al., 1999). The principal challenge with the shape-preserving and monotonic approaches appears to be to decide how to precisely determine the required attributes of the interpolator, and how to tailor it to respect them, since there is no universal best choice.

2.5 **Stability and accuracy**

Analyses of the stability properties of the semi-Lagrangian advection scheme (e.g. Bates and McDonald, 1982; McDonald, 1984; Pudykiewicz and Staniforth, 1984) show that the maximum timestep is not limited by the maximum wind speed, as is the case for explicit Eulerian advection schemes, and consequently it is possible to stably integrate with Courant numbers ($C = U \Delta t / \Delta x$) that far exceed unity.

In general it is found that semi-Lagrangian advection is competitive with Eulerian advection with respect to accuracy, but it has the added advantage that this accuracy can be achieved at less computational cost, since models can be integrated stably with timesteps that far exceed the maximum-possible timesteps of Eulerian schemes. The aforementioned stability analyses show that semi-Lagrangian advection schemes have very good phase speeds with little numerical dispersion, but contrary to some Eulerian schemes (e.g. leapfrog-based schemes) there is some damping due to interpolation as discussed in Section 2.4. This damping is fortunately very scale selective (at least when using high-order interpolators). McCalpin (1988) has theoretically compared this damping with more traditional forms such as Laplacian and biharmonic dissipation, and derived some criteria to ensure that the damping due to semi-Lagrangian advection is less than that due to the more traditional forms.

Semi-Lagrangian advection is intimately connected with other advection methods that have appeared in the literature over the years, including particle-in-cell (e.g. Raviart, 1985) and characteristic Galerkin (e.g. Morton, 1985) methods. Indeed for uniform advection in 1-d, the simplest semi-Lagrangian advection scheme (using linear interpolation, and not recommended) is equivalent to both classical upwinding and to the simplest characteristic Galerkin method; and semi-Lagrangian advection using cubic-spline interpolation is equivalent to the higher-order characteristic Galerkin methods of Morton (1985) and also to a particle-in-cell method described in Eastwood (1987).

Several well-known Eulerian methods can also be interpreted as being special cases of semi-Lagrangian ones. Thus the Lax and Wendroff (1960), Takacs (1985) 3rd-order, and Tremback et al. (1987) schemes are respectively equivalent for 1-d uniform advection to semi-Lagrangian schemes with quadratic-Lagrange, cubic-Lagrange and n-th-order-Lagrange interpolation. Note however that these Eulerian methods are restricted to Courant numbers less than unity and are consequently less general than their semi-Lagrangian counterparts.

Although the semi-Lagrangian method is equivalent for uniform 1-d advection to several other methods, what distinguishes it from other methods is that it generalizes differently to non-uniform advection in multi-dimensions. The principal difference is the use of (10), introduced in Robert (1981), for the trajectory calculations. Of particular importance is that *the approximation of the trajectory equation (8) is* $O(\Delta t^2)$ *accurate*. It is possible to use a simpler, and cheaper, $O(\Delta t)$ accurate method to approximate the displacement equation (8) but this can dramatically deteriorate the accuracy of the scheme, as shown by Staniforth and Pudykiewicz (1985) and Temperton and Staniforth (1987), and analysed by McDonald (1987). Consequently most semi-Lagrangian schemes use an $O(\Delta t^2)$ method for discretising the trajectory equation.

3. APPLICATION TO COUPLED SETS OF EQUATIONS

To illustrate how semi-Lagrangian advection can be advantagously used to solve coupled systems of equations, we describe its application to the discretisation of the shallow-water equations

$$\frac{dU}{dt} + \phi_x - fV = 0, \tag{14}$$

$$\frac{dV}{dt} + \phi_y + fU = 0, \tag{15}$$

$$\frac{d\ln\phi}{dt} + U_x + V_y = 0, \tag{16}$$

where U and V are the wind components, ϕ ($=gz$) is the geopotential height (i.e. height multiplied by g) of the free surface of the fluid above a flat bottom, and f is the Coriolis parameter.

These equations are often used in NWP to test new numerical methods, since they are a 2-d prototype of the 3-d equations that govern atmospheric motions. They share several important properties with their progenitor. A linearisation of the equations reveals that there are two basic kinds of associated motion: slow-moving Rossby modes (which most affect the large-scale weather motions and, to leading order, move at the local wind speed) and small-amplitude fast-moving gravitational oscillations (which are inadequately represented at initial time due to the paucity of the observational network). From a numerical standpoint this has the important implication that

the timestep of an explicit Eulerian scheme (e.g. leapfrog) is limited by the speed of the fastest-moving gravity mode. Since for atmospheric motions this speed is much faster than those associated with the Rossby modes that govern the weather, this leads to timesteps that are much shorter than those associated with an explicit treatment of advection. A time-implicit treatment of the pressure-gradient term of the vector momentum equation [2nd terms of (14) and (15)] and horizontal divergence of the continuity equation [2nd and 3rd terms of (16)], introduced in Robert (1969) and termed the *semi-implicit* scheme, allows stable integrations with no loss of accuracy using timesteps that are much longer than that of the leapfrog scheme. The price to be paid for this increase in timestep length is the need to solve an elliptic-boundary-value problem once per timestep: nevertheless this improves efficiency by approximately several factors. Analysis shows that the maximum-possible timestep length is then limited by the Eulerian treatment of advection.

Early applications of semi-Lagrangian advection to coupled sets of equations (e.g. Krishnamurti, 1962, 1969; Mathur, 1970, 1974) did not take advantage of the enhanced stability properties of the method, since the models were formulated in such a way that they were not, in the terminology of Bates and McDonald (1982), "multiply upstream" and so the Courant number (associated with the treatment of advection) was always less than unity. Nevertheless these studies did demonstrate that semi-Lagrangian advection is an acceptably-accurate method for advection. Robert (1981, 1982) reasoned that since semi-Lagrangian advection is stable for Courant numbers significantly larger than unity, it should be possible to associate a semi-Lagrangian treatment of advection with a semi-implicit treatment of the terms responsible for gravitational oscillations. This was done in the context of a three-time-level scheme. A stability analysis was given to demonstrate that this scheme should be stable with timesteps that exceed those of the gravitational, advective and inertial limits, and this was verified in sample integrations.

To illustrate the application of the semi-Lagrangian method we discretise the shallow-water equations using a *two*-time-level semi-implicit semi-Lagrangian scheme, which permits a further doubling of efficiency with respect to the Robert (1982) algorithm at no extra cost. For simplicity we describe the scheme in plane geometry. It is then formally equivalent to that of Temperton and Staniforth (1987) with the map-scale factor set to unity. In spherical geometry the discretisation is a little more complicated due to the appearance of metric terms in the momentum equations. These can be trivially absorbed into the formulation given below, using either the approach of Ritchie (1988) or that of Côté (1988) and Bates et al. (1990). Thus

$$\frac{U^+ - U^0}{\Delta t} + \frac{\phi_x^+ + \phi_x^0}{2} - \frac{1}{2}\left[(fV)^+ + (fV)^0\right] = 0, \tag{17}$$

$$\frac{V^+ - V^0}{\Delta t} + \frac{\phi_y^+ + \phi_y^0}{2} + \frac{1}{2}\left[(fU)^+ + (fU)^0\right] = 0, \tag{18}$$

$$\frac{\ln\phi^+ - \ln\phi^0}{\Delta t} + \frac{1}{2}\left[\left(U_x + V_y\right)^+ + \left(U_x + V_y\right)^0\right] = 0, \tag{19}$$

where (14)-(16) have been discretised using (11) with R set to zero. Here advection terms are treated as time-differences along the trajectories and all other terms are treated as time-averages along the trajectories, leading to an $O(\Delta t^2)$-accurate scheme. Where traditional (three-time-level) semi-implicit time discretisations have an

explicit time-treatment of the Coriolis terms, the above discretisation employs a time-implicit treatment (as in Robert, 1982) in order to achieve an $O(\Delta t^2)$-accurate scheme: note that explicitly evaluating these terms at time t would not only reduce the accuracy to $O(\Delta t)$ but would also lead to instability. The trajectories are computed using the discretised equations (12)-(13) introduced by Temperton and Staniforth (1987) and McDonald and Bates (1987).

For the 1-d shallow-water equations it can be shown that there are three characteristic velocites in the coupled set, one being the local wind speed and associated with the slow Rossby modes that govern weather motions, the other two being associated with the propagation of gravitational oscillations. Thus the coupling of a semi-Lagrangian treatment of advection with a semi-implicit treatment of gravitational oscillations corresponds to integrating along the most important characteristic direction of the problem (i.e. that associated with the local windspeed): this is somewhat similar in spirit to a suggestion given on p. 860 of Morton (1985).
Eqs. (17)-(18) can be manipulated to give

$$U^+ = -\frac{\Delta t}{2}\left[a\phi_x^+ + b\phi_y^+\right] + known, \tag{20}$$

$$V^+ = -\frac{\Delta t}{2}\left[a\phi_y^+ - b\phi_x^+\right] + known, \tag{21}$$

where $a = \left[1 + (f\Delta t/2)^2\right]^{-1}$ and $b = (f\Delta t/2)a$. Taking the divergence of (20)-(21) and eliminating this in (19) then leads to the elliptic-boundary-value problem

$$\left[(a\phi_x)_x + (a\phi_y)_y + (b\phi_y)_x - (b\phi_x)_y - 4\frac{\ln\phi}{\Delta t^2}\right]_{(x,t+\Delta t)} = known. \tag{22}$$

We now summarise the above as the following algorithm:

(i) Extrapolate $\mathbf{V}$ using (13) and solve (12) iteratively for the displacements α_m for all meshpoints x_m, using values at the previous timestep as initial guess, and an interpolation formula. Note that it is only necessary to perform this computation once per timestep, since the same trajectory is used for all three advected quantities.

(ii) Compute upstream (superscript 0) quantities in (17)-(19) by first computing derivative terms (e.g. U_x) and *then* evaluating quantities upstream (these two operations are *not* commutative!). Here it is more efficient to collect together all terms to be evaluated upstream in a given equation before interpolating (the distributive law applies).

(iii) Solve the elliptic-boundary-value problem (22) for $\phi(x,t+\Delta t)$.

(iv) Back substitute $\phi(x,t+\Delta t)$ into (20)-(21) to obtain $U(x,t+\Delta t)$ and $V(x,t+\Delta t)$.

The above elliptic-boundary-value problem is weakly non-linear and is solved iteratively using ϕ at the previous timestep as a first guess. It is only marginally more expensive to solve than the Helmholtz problem associated with traditional three-time-level semi-implicit Eulerian discretisations.

Semi-Lagrangian advection has also been successfully coupled with the split-explicit method (Bates and McDonald, 1982) and the alternating-direction-implicit method (Bates, 1984; Bates and McDonald, 1987). Both

of these approaches have the virtue of being simpler than the semi-implicit semi-Lagrangian one (there is no elliptic-boundary-value problem), but unfortunately they do not perform as well, particularly at large timestep. To date it appears that the best schemes arise from associating semi-Lagrangian advection with a semi-implicit scheme, and that timesplitting is best avoided since it introduces unacceptably-large truncation errors for large timesteps.

4. SOME EARLY ADVANCES

When Robert (1981) proposed associating a semi-Lagrangian treatment of advection with a semi-implicit treatment of gravitational oscillations, it was thought that this approach was restricted to three-time-level schemes in Cartesian geometry using a finite-difference discretisation. This has happily proved not to be the case, and in this section we discuss some important extensions of the approach. Although important, the extension to *two-time-level schemes* has already been discussed in some detail, and will therefore only be briefly discussed in this section in the context of other extensions. Only early extensions of the semi-Lagrangian methodology are given here. This is both for reasons of space, and because more recent advances are in any case covered in other contributions to this volume.

4.1 Finite-element discretisations and variable-resolution

Pudykiewicz and Staniforth (1984) coupled semi-Lagrangian advection with a uniform-resolution finite-element discretisation of the diffusion terms in the solution of the 2-d advection-diffusion equation, and this was extended to the 3-d case in Pudykiewicz et al. (1985). Staniforth and Temperton (1986) extended the methodology in the context of a coupled system of equations (the shallow-water equations) in two ways. Firstly they showed that in this context the semi-Lagrangian method can be coupled to a spatial discretisation scheme other than a finite-difference one, viz. a *finite-element* discretisation, and secondly that it can also be applied on a *variable-resolution* Cartesian mesh. A further doubling of efficiency was then demonstrated in Temperton and Staniforth (1987) by replacing the three-time-level scheme of the Staniforth and Temperton (1986) model with a two-time-level one.

4.2 Non-interpolating schemes

The interpolation in a semi-Lagrangian scheme, as mentioned previously, leads to some damping of the smallest scales. To address this problem Ritchie (1986) proposed a non-interpolating version of semi-Lagrangian advection. The basic idea here is to decompose the trajectory vector into the sum of two vectors, one of which goes to the nearest meshpoint, the other being the residual. Advection along the first trajectory is done via a semi-Lagrangian technique that displaces a field from one meshpoint to another (and therefore requires no interpolation), while the advection along the second vector is done via an undamped three-time-level Eulerian approach such that the residual Courant number is always less than one. Thus the attractive stability properties of interpolating semi-Lagrangian advection are maintained but without the consequent damping. The non-interpolating methodology is not restricted to gridpoint discretisations and has also been successfully applied to spectral discretisations (Ritchie, 1988).

An alternative way of viewing the non-interpolating formalism of Ritchie (1986) is presented in Smolarkiewicz and Rasch (1990). They showed that it is possible to convert any advection algorithm into a semi-Lagrangian framework, thus permitting the use of much larger timesteps with the scheme for little additional cost. This interesting realisation is of potential benefit for models whose maximum timestep is limited by an Eulerian treatment of advection. To demonstrate this idea they successfully extended the stability limit of the Tremback et al. (1987) family of algorithms. In so doing they obtained a family of schemes which is equivalent to using a time-split semi-Lagrangian scheme with Lagrange interpolation.

4.3 Spherical geometry

The convergence of the meridians at the poles of an Eulerian finite-difference model in spherical geometry leads to unacceptably-small timesteps being required in order to maintain computational stability. The usual approach to this problem is to somehow filter the dependent variables in the vicinity of the poles. While this procedure does relax the stability constraint, it unfortunately deteriorates accuracy. Ritchie (1987) demonstrated that it is possible to passively advect a scalar over the pole using semi-Lagrangian advection with timesteps far exceeding the limiting timestep of Eulerian advection schemes. This paved the way to applications in global *spherical* geometry. The first such application was to couple semi-Lagrangian advection with a *spectral* representation (i.e. expansion in terms of spherical harmonics) of the dependent variables to solve the shallow-water equations over the sphere (Ritchie, 1988). A new problem arose here associated with the stable advection of a *vector* quantity (momentum). The solution proposed in Ritchie (1988) is to introduce a tangent plane to avoid a weak instability due to a metric term. The diagnosis of this problem, which led to the tangent plane algorithm, is described in Desharnais and Robert (1990). Côté (1988) and Bates et al. (1990) have respectively proposed alternative approaches based respectively on the use of Lagrange multipliers and the vector form of the momentum equation. Although these three approaches appear very different, they nevertheless lead to essentially the same algorithm.

Ritchie (1988) successfully integrated his shallow-water model with a timestep six-times longer than that of the limiting timestep of the corresponding Eulerian semi-implicit spectral model (which in turn uses a six-times-longer timestep than that of an Eulerian leapfrog model). Côté and Staniforth (1988) then further doubled the efficiency of the Ritchie (1988) model, by replacing its three-time-level scheme by a two-time-level one analogous to that of Temperton and Staniforth (1987) for Cartesian geometry.

4.5 3-d applications

Thus far we have mostly discussed the use of semi-Lagrangian advection for extending the limiting timestep of 2-d applications for NWP. To be useful the method must also be applicable in 3-d. A first step in this direction was taken in Bates and McDonald (1982), where a semi-Lagrangian treatment of *horizontal* advection in a 3-d (baroclinic primitive equations) model was coupled with a split-explicit time scheme in the Irish Meteorological Service's operational model of the time. This was the first scheme to demonstrate the enhanced stability of semi-Lagrangian advection in a 3-d model, and the first to be used operationally. However it is only $O(\Delta t)$ accurate and although stable with long timesteps, the increase in timestep is consequently very much limited by accuracy considerations. Robert et al. (1985) then introduced a three-time-level $O(\Delta t^2)$-*accurate* 3-d limited-

area gridpoint model with a semi-Lagrangian treatment of *horizontal* advection, and were able to successfully integrate with longer timesteps than had hitherto been possible. This demonstrated the practical importance of achieving $O(\Delta t^2)$ accuracy.

A somewhat similar model to the Robert et al. (1985) one, but with mountains included, is described in Kaas (1987). It was reported that when strong winds blow over steep mountains, instabilities may appear if the linear part [$(\phi + RT_0 \ln p_s)$] of the horizontal pressure gradient term in sigma coordinates is evaluated as the average of values at the endpoints $[(x, t+\Delta t), (x - 2\alpha, t-\Delta t)]$ of the trajectory, but the nonlinear part $[R(T-T_0)\ln p_s]$ is evaluated at the midpoint $(x - \alpha, t)$. This behaviour was attributed to a lack of balance (in the discrete approximation) between two large terms of opposite sign, due to their being evaluated at different geographical points. The reported solution to this problem is to evaluate the non-linear part as the average of its values at the geographical points associated with arrival (x) and departure $(x - 2\alpha)$, both values being taken at the intermediate time level t. While this approach appreciably mitigates the problem, it does not resolve it completely. Coiffier et al. (1987) studied it in the context of a 2-d linearised baroclinic model, and show that the use of semi-Lagrangian advection with *large* timesteps leads to an incorrect steady-state solution when the model is orographically forced. Their analysis to explain this behaviour also applies to the formulation proposed by Kaas (1987). It suggests that the seriousness of the problem is a function of timestep, windspeed and detail (the larger the timestep and windspeed, and the more detailed the orography, the worse is the problem), and of whether the time scheme is a two- or three-time-level one (two-time-level schemes are better since the problem first occurs with timesteps twice as long as those of three-time-level schemes), and this was confirmed by the Rivest et al. (1993) analysis . They showed that a centred semi-implicit semi-Lagrangian scheme gives rise to a spurious numerical orographic resonance, and that this can be addressed by a time decentring of the scheme.

The timesteps of the early 3-d semi-implicit semi-Lagrangian models are limited by the stability of an explicit Eulerian treatment of *vertical* advection: or put another way, vertical resolution is limited when using a large timestep. To remove this limitation, Tanguay et al. (1989) proposed a three-time-level model that uses semi-Lagrangian advection in all three space dimensions.

4.6 **Non-hydrostatic systems**

As computers become ever more powerful, it becomes possible to run models at higher and higher resolution. This motivates the need to *efficiently* integrate non-hydrostatic systems of equations for real-time forecasting applications over large domains. Such systems admit *acoustic* modes, which travel much faster than either Rossby or gravity modes. Consequently if care is not exercised, the limiting timestep will be even more restrictive than that associated with an explicit primitive equations model. This is because an explicit time treatment of the terms associated with the propagation of acoustic energy leads to a limiting timestep that is much smaller than that associated with an explicit treatment of gravity-wave terms, completely eliminating the efficiency advantage of a semi-implicit/ semi-Lagrangian treatment of the gravity/ Rossby mode terms.

Since the acoustic modes carry very little energy, it is permissible to slow them down by the use of a time-implicit treatment of the terms responsible for their existence, by analogy with the retarding of the gravity modes

by the semi-implicit scheme. This is the approach taken in Tanguay et al. (1990), who generalise the semi-implicit semi-Lagrangian methodology for the hydrostatic primitive equations to the non-hydrostatic case. They show that it is possible to integrate the fully-compressible non-hydrostatic equations for little additional cost, and this opened the way to highly-efficient non-hydrostatic models.

5. CONCLUSIONS

The semi-Lagrangian methodology has been presented together with some of the early advances. Information on more recent advances may be found in companion papers in this volume.

Since André Robert's pioneering work of the early eighties, the semi-implicit semi-Lagrangian approach has been extended from finite-difference applications in Cartesian geometry to finite-difference, finite-element and spectral applications in both Cartesian and spherical geometry. Best results have generally been obtained when coupling semi-Lagrangian advection to a semi-implicit treatment of gravitational oscillations, rather than to splitting methods such as split-explicit and alternating-direction-implicit. It is crucial to avoid introducing $O(\Delta t)$ truncation errors in either the trajectory computations or the discretization of the governing equations, in order to fully reap the benefit (i.e. long timesteps) of enhanced stability. It is also important to use the semi-Lagrangian method for vertical as well as for horizontal advection, in order to avoid unduly limiting vertical resolution. This has the added benefit of facilitating a higher-order accurate treatment of vertical advection.

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