Random version of the Benjamin-Feir Instability

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1. Introduction

This is a review of my attempt, from the 1980's, to try to understand why the Benjamin-Feir instability, which is well-known to exist in wave tank experiments(cf. Benjamin&Feir, 1967), has no equivalent in the theory of nonlinear four-wave interactions of a homogeneous, random wave field. It turns out that there is simply no Benjamin-Feir instability (others prefer to use the term modulational instability or side-band instability) for a homogeneous sea. In order to capture this instability in a random sea, inhomogeneities in the wave field should be allowed for. In particular, for a detailed evolution of the wave field on a fine mesh grid the Benjamin-Feir instability may therefore play a role.

Presently, there is considerable interest in extreme sea states as they occur with freak waves, rogue waves and the "three sisters"(a group of three fairly large waves). Since the work of Lake et al (1977) it is well-known that the Benjamin-Feir instability causes a nearly uniform wave to evolve towards a strongly modulated state consisting of a number of wave groups. Therefore, the Benjamin-Feir instability could play an important role in the generation of extreme events such as freak waves, and here it is briefly discussed under what conditions this may happen.

2. Benjamin-Feir Instability in an inhomogeneous sea

Since the fundamental investigations of Phillips (1960) and Hasselmann(1962) there has been a considerable interest in the energy transfer due to four-wave interactions of a homogeneous, random sea. The effect of nonlinear interactions is twofold. First, four-wave interactions are controlling the shape of the high-frequency part of the spectrum, and second, they are important in shifting the peak of the spectrum towards lower frequencies. For these reasons the discrete interaction approximation to four-wave interactions was implemented by the Wave Model Development and Implementation(WAMDI) group(1988) in the third generation wave prediction model WAM.

It is important to discuss the range of validity of the statistical theory of four-wave interactions of a homogeneous wave field (Janssen,1991). It can be estimated that the resulting nonlinear energy transfer occurs on the long time scale $T_{NL} = 1/(\varepsilon^4\omega_0)$, since the rate of change in time of the action density N is proportional to N^3. Here, ε is a typical wave steepness and ω_0 is a typical frequency of the wave field, e.g. the peak frequency. The theory is formally only valid for weakly nonlinearwaves ($\varepsilon \ll 1$) because of the assumption of (nearly) Gaussian statistics. An additional restriction of the nonlinear theory of a homogeneous wave field is that the corresponding frequency spectrum is broad enough; that is, the spectral width σ should satisfy the inequality sigma > $\varepsilon\omega_0$. This last condition can only be understood in the framework of the inhomogeneous theory of four-wave interactions.

The Hasselmann theory of four-wave interactions gives an adequate description of the slow time and space evolution of a random, homogeneous wave field. On a short time or spatial scale, another description of the wave field is more appropriate. And indeed, a much faster energy transfer is possible in the presence of spatial inhomogeneities. For an inhomogeneous, random sea, Alber and Saffman (1978) and Alber(1978) derived an equation describing the evolution of a narrowband wave train.In addition, Crawford et al (1980), following Zakharov's approach obtained a unified equation for the evolution of a random field of deep water waves, which accounts for both the effects of spatial inhomogeneities and the energy transfer associated with the homogeneous spectrum. From this analysis it became apparent that inhomogeneities give rise to a much faster energy transfer, $T_{NL} = 1/(\varepsilon 2\omega_0)$, comparable with the typical time scale of the Benjamin-Feir instability.

Alber and Saffman(1978) found the important result that inhomogeneities in a homogeneous wave field were generated by an instability (which I call the random version of the Benjamin-Feir instability), provided the width of the spectrum is sufficiently small. For a spectrum with a Gaussian shape, instability was found if sigma < $\varepsilon\omega_0$. In the limit of vanishing width, the deterministic results of Benjamin and Feir (1967) on the instability of a uniform wave train were rediscovered. Clearly, finite band width is stabilizing. The stability criterion, $\sigma > \varepsilon\omega_0$, tells us that the growth rate of the Benjamin-Feir instability vanishes as the correlation time of the random wave field (about $1/\sigma$) is reduced to the order of the characteristic time scale for modulational instability (about $2\pi/\omega_{max}$, where at modulation frequency ω_{max} one has maximum growth for $\sigma \rightarrow 0$). Thus, de-correlation of the phases leads, as expected, to stabilization of the wave train on the short time scale $T_{NL} = 1/(\varepsilon^2 \omega_0)$, and nonlinear energy transfer is then only possible on the much longer time scale $T_{NL} = 1/(\varepsilon^4 \omega_0)$.

The result of Alber and Saffman has the following implications:

(1) for sufficiently broad spectra only Hasselmann's four-wave interactions are relevant.

(2) spectra derived from sufficiently long time series should have a width larger than $\varepsilon\omega_0$, because for smaller spectral width the spectrum would be unstable and the Benjamin-Feir instability would generate sidebands leading to a broadening of the spectrum (Janssen, 1983).

(3) since the spectral width exceeds a minimum value, the average length of a wave group $<\ell>$ must be smaller than some maximum. For a steepness $\varepsilon = 0.1$ theory tells us that $<\ell_{max} = 2$.

Janssen and Bouws(1986) (for an easier accesible reference see Janssen (1991)) tested the conjecture (2) by using spectra from weather station IJmuiden. In order to get a high enough statistical significance spectra were obtained from time series that were at least 20 minutes long. It was found that the observed spectra indeed obeyed the condition for a stable wave train, $\sigma > \varepsilon\omega_0$. This provides justification for the use of Hasselmann's four-wave interactions in ocean wave prediction systems, if one is interested in relatively large scale, slow time applications. However, it should be emphasized that if one is interested in questions such as the probability of the occurrence of extreme phenomena such as the "three Sisters", which may involve relatively short spatial and time scales, it seems very likely that inhomogeneous nonlinear effects should be taken into account. For one thing, the Benjamin-Feir Instability may be relevant in focussing considerable amounts of wave energy in one location, thus producing three Sisters-like events.

3. Conclusion

We have discussed the possible relevance of the Benjamin-Feir Instability for generation of extreme events. The Benjamin-Feir instability normally operates on a very short time scale resulting in a rapid broadening of the wave spectrum until the instability is quenched. For large scale application the Benjamin-Feir instability is not relevant. However, if one is interested in extreme events which may be rather short-lived, nonlinear inhomogeneous interactions related to the Benjamin-Feir instability are expected to be relevant.

Because the inhomogeneous nonlinear interactions are so rapid, it seems to me that it is possible to develop a diagnostic tool to provide information on the probability of extreme events. Since the usual wave prediction systems describe the slow evolution of the wave spectrum, it must be possible, for given large scale spectrum(which is supposed to describe the averaged sea state over a large area) to determine the probability of occurence of extreme wave height using a simple model such as the nonlinear Schrodinger equation. In other words, starting from initial conditions for the surface elevation, obtained from the given spectrum with random phases, one needs to perform ensemble integrations with the nonlinear Schrodinger equation to obtain the probability distribution function(pdf) for maximum waveheight. It will be of interest to see to what extent the thus found pdf differs from the standard Rayleigh distribution.

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