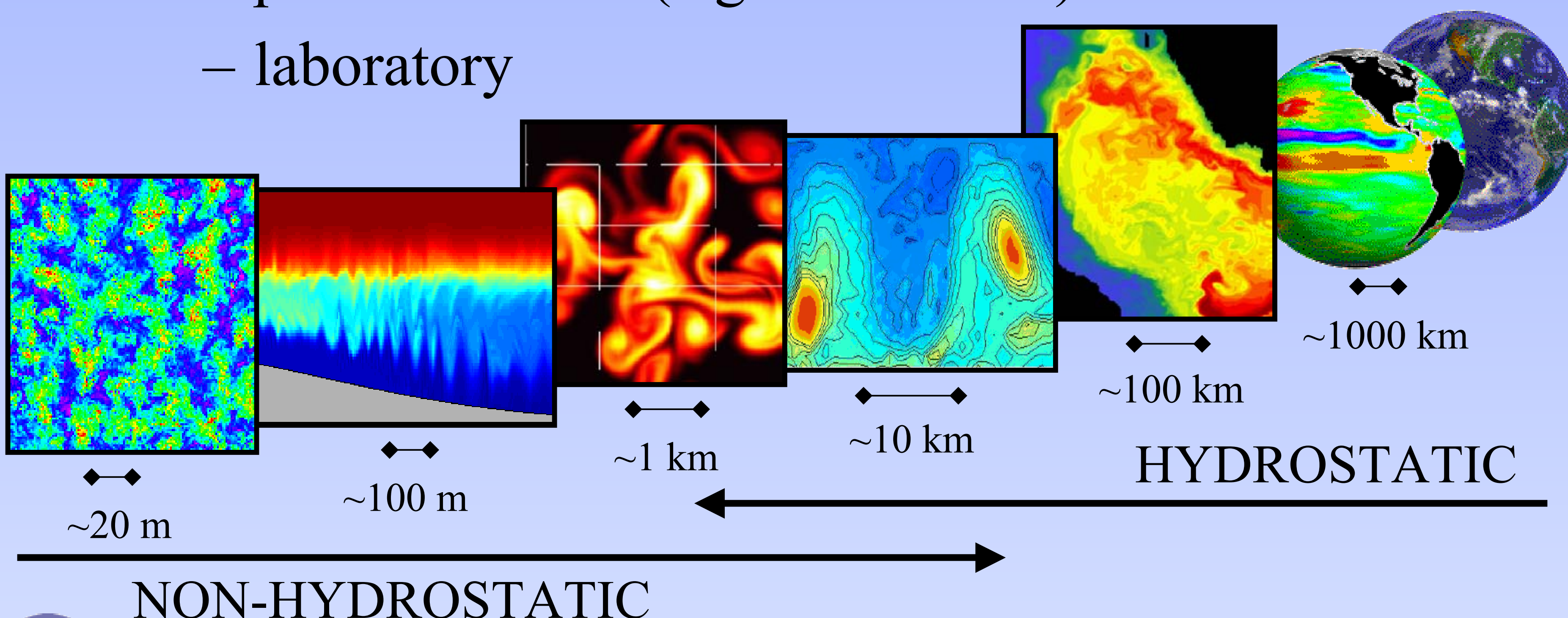
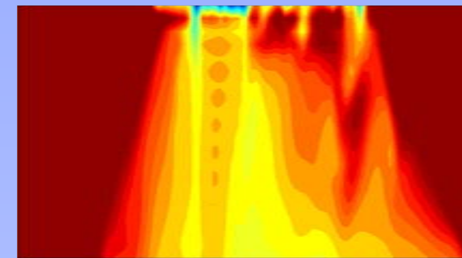
Formulation and numerics of the MITgcm

- Model for all scales
  - large scale circulation and regional models
  - process studies (e.g. convection)
  - laboratory

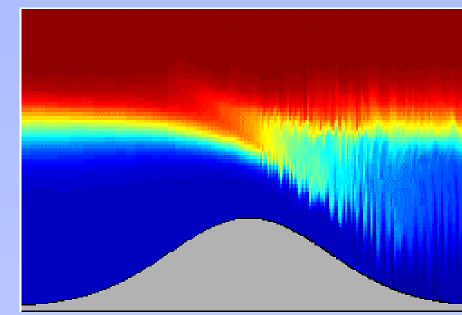


MITgcm features I

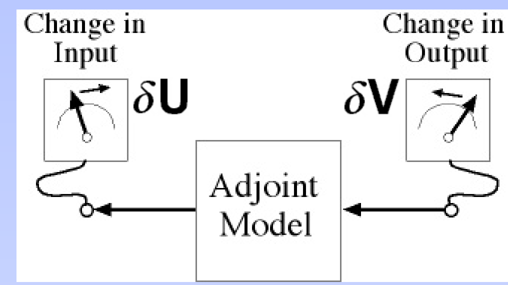
- Non-hydrostatic and quasi-hydrostatic
  - Resolve mixing processes



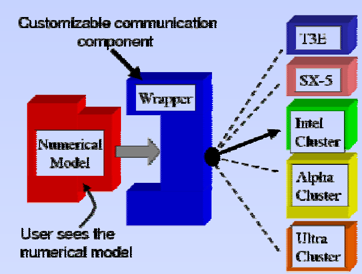
- Finite volume method/shaved cells
  - Accurate representation of topography



- Automatic adjoint (TAMC/TAF FastOpt)
  - ECCO project (state estimation, D.A., …)

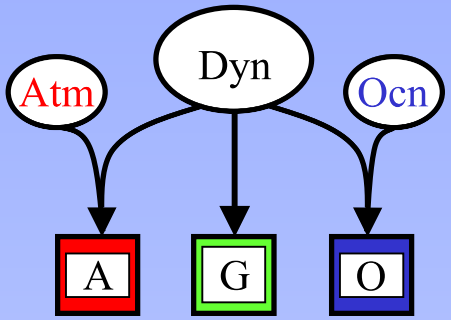


- Computing software/technology
  - Personal super-computing, wide portability, …

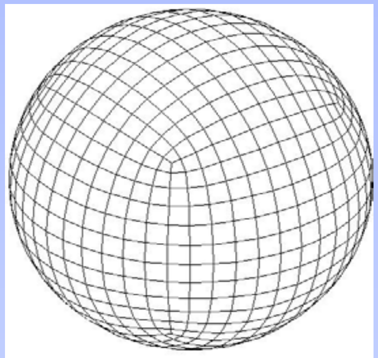


MITgcm features II

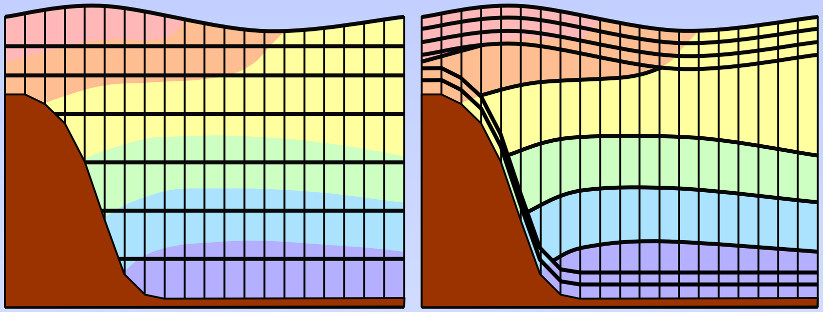
- Isomorphisms
  - Ocean ↔ Atmosphere
  - Boussinesq ↔ non-Boussinesq



- Gridding the sphere
  - Expanded spherical cube, finite volume/vector invariant



- Vertical coordinates
  - New coordinates
  - New class of model



Model Equations

$$\rho_o D_t \vec{v} + 2\Omega \times \rho_o \vec{v} + g\rho\hat{k} + \nabla p = \vec{F}$$

- Momentum

$$\rho_o \nabla \cdot \vec{v} = 0$$

- Continuity

$$\partial_t \eta + \nabla \cdot (H + \eta)\vec{v}_h = P - E$$

- Free-surface

$$D_t \theta = Q_\theta$$

- Internal energy

$$D_t s = Q_s$$

- Salt

$$\rho = \rho(s, \theta, p)$$

- Equation of state

- Boussinesq *(in height coordinates)*
  - linearizes momentum $\quad \rho' = (\rho - \rho_o) << \rho_o, \quad \rho\vec{v} \rightarrow \rho_o \vec{v}$
- Incompressible
  - conserves volume $\quad D_t \rho << \rho \nabla \cdot \vec{v}$
  - filters out acoustic modes
- Non-hydrostatic
  - deep/shallow atmosphere approx.

$C_s \sim 1500$ m s^{-1}

$\sqrt{gH} \sim 150$ m s^{-1}

$NH \sim 3$ m s^{-1}

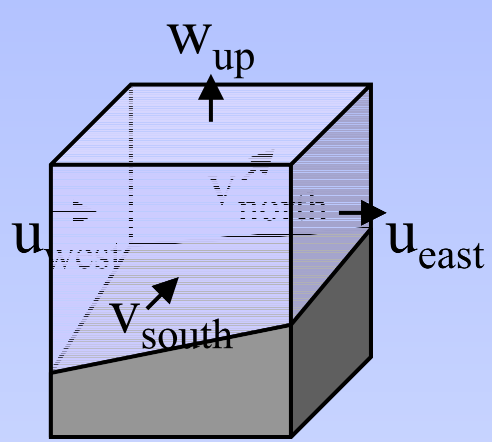
$U \sim 1$ m s^{-1}

Andy White

Finite volume formulation

- Continuity equation: $\nabla \cdot \vec{v} = 0 \rightarrow \int \vec{v} \cdot d\vec{A} = 0$
- Irregular domain → grid point model
- "C" grid/Lorenz (natural)

Shaved/cut cells



Volume budget:

$$\begin{aligned} & A^{u}_{east} u_{east} - A^{u}_{west} u_{west} \\ & + A^{v}_{north} v_{north} - A^{v}_{south} v_{south} \\ & + A^{w}_{up} w_{up} - A^{w}_{down} w_{down} = 0 \end{aligned}$$

$$\vec{v} \cdot \hat{n} = 0 \Rightarrow \quad A^{w}_{down} w_{down} = 0$$

- Tracer equation discretized consistently

Core Algorithm: Projection method

Chorin, 1968
Miller, 1974

- Discretize momentum in time:

$$\rho_o \vec{v}^{n+1} + \Delta t \nabla p = \rho_o \vec{v}^{n} + \Delta t \vec{G} \qquad = \rho_o \vec{v}^{*}$$

No time-level for "p"

- Substitute into continuity:

$$\delta_i\left(A^u u^{n+1}\right) + \delta_j\left(A^v v^{n+1}\right) + \delta_k\left(A^w w^{n+1}\right) = 0$$

- 3D Elliptic equation for pressure:

$$\boxed{\frac{\Delta t}{\rho_o} \nabla^2 p = \nabla \cdot \vec{v}^{*}}$$

  - 7 point stencil

$$\boxed{\begin{aligned}\frac{\Delta t}{\rho_o}\left[\delta_i\left(\frac{A^u}{\Delta x}\delta_i p\right) + \delta_j\left(\frac{A^v}{\Delta y}\delta_j p\right) + \delta_k\left(\frac{A^w}{\Delta z}\delta_k p\right)\right] \\ = \delta_i\left(A^u u^*\right) + \delta_j\left(A^v v^*\right) + \delta_k\left(A^w w^*\right)\end{aligned}}$$

  - expensive to solve in irregular domain on grid points

Note: B.C.'s equivalent to $\Delta t \nabla p \cdot \hat{n} = \rho_o \vec{v}^{*} \cdot \hat{n}$

Efficient N-H modeling

- Partition the pressure:

Surface "Hydrostatic" Non-hydrostatic

$$p = \underbrace{p_s(x,y) + p_h(x,y,z)}_{p_H} + p_{nh}(x,y,z)$$

- Surface pressure (2D):

$$\nabla_z \cdot H \nabla_z p_s = \frac{\rho_o}{\Delta t} \nabla \cdot \int_{-H}^{0} (\vec{v}_h^{\,*} - \Delta t \nabla_z p_h - \cancel{\Delta t \nabla_z p_{nh}})\, dz$$

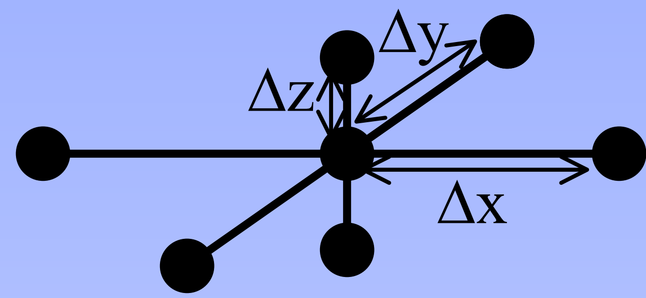
neglect here

- Non-hydrostatic pressure (3D):

$$\nabla^2 p_{nh} = \frac{\rho_o}{\Delta t} \nabla \cdot (\vec{v}^{\,*} - \Delta t \nabla_z p_s - \Delta t \nabla_z p_h)$$

Efficient N-H modeling II

- Small aspect ratio
  - "stiff" problem



$$\nabla_h^2 \mathrm{p}_{\mathrm{nh}} + \partial_z^2 \mathrm{p}_{\mathrm{nh}} \approx \frac{1}{\Delta z^2}\left[\frac{\Delta z^2}{\Delta x^2}\delta_{ii} + \frac{\Delta z^2}{\Delta y^2}\delta_{jj} + \delta_{kk}\right]\mathrm{p}_{\mathrm{nh}}$$

Dominant term in operator

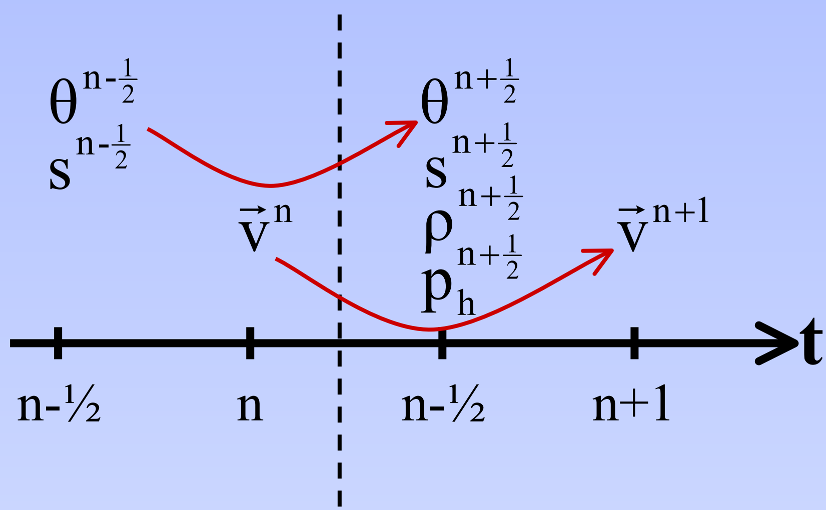
- Pre-conditioner builds on aspect ratio:

$$\nabla_h^2 \mathrm{p}_{\mathrm{nh}} + \partial_z^2 \mathrm{p}_{\mathrm{nh}} \approx \frac{1}{\Delta z^2}\delta_{kk}\mathrm{p}_{\mathrm{nh}}$$

  - tri-diagonal in vertical direction ⇒ *cheap*
  - parallel decomposition in horizontal only

[Staggered] Algorithm

- Stagger variables in time
  - "leap-frog" without computational mode
  - 2^{nd} order explicit gravity waves
  - centered **advecting** flow



Arbitrary time-stepping

$$\theta^{n+\frac{1}{2}} = \theta^{n-\frac{1}{2}} + \Delta t\left(Q_\theta{}^{n} - \nabla\cdot F(\vec{v}^{n},\tilde{\theta})\right)$$

$$s^{n+\frac{1}{2}} = s^{n-\frac{1}{2}} + \Delta t\left(Q_s{}^{n} - \nabla\cdot F(\vec{v}^{n},\tilde{s})\right)$$

$$\rho^{n+\frac{1}{2}} = \rho\left(s^{n+\frac{1}{2}},\theta^{n+\frac{1}{2}},p_o(z)\right)$$

$$p_h^{n+\frac{1}{2}} = -\int_z^0 g\rho^{n+\frac{1}{2}}dz'$$

Projection method

$$\vec{v}^* = \vec{v}^{n} + \Delta t\left(G(\tilde{\vec{v}}) + F^{n+\frac{1}{2}} - \nabla_z p_h^{n+\frac{1}{2}}\right)$$

$$\nabla^2 p_{nh} = \nabla\cdot\vec{v}^*$$

$$\vec{v}^{n+1} = \vec{v}^* - \Delta t\nabla p_{nh}$$

Free surface in height coordinates

- Rigid-lid
  - Poisson eq$^{\underline{n}}$ for p_s

$$\nabla \cdot H\vec{v} = 0$$

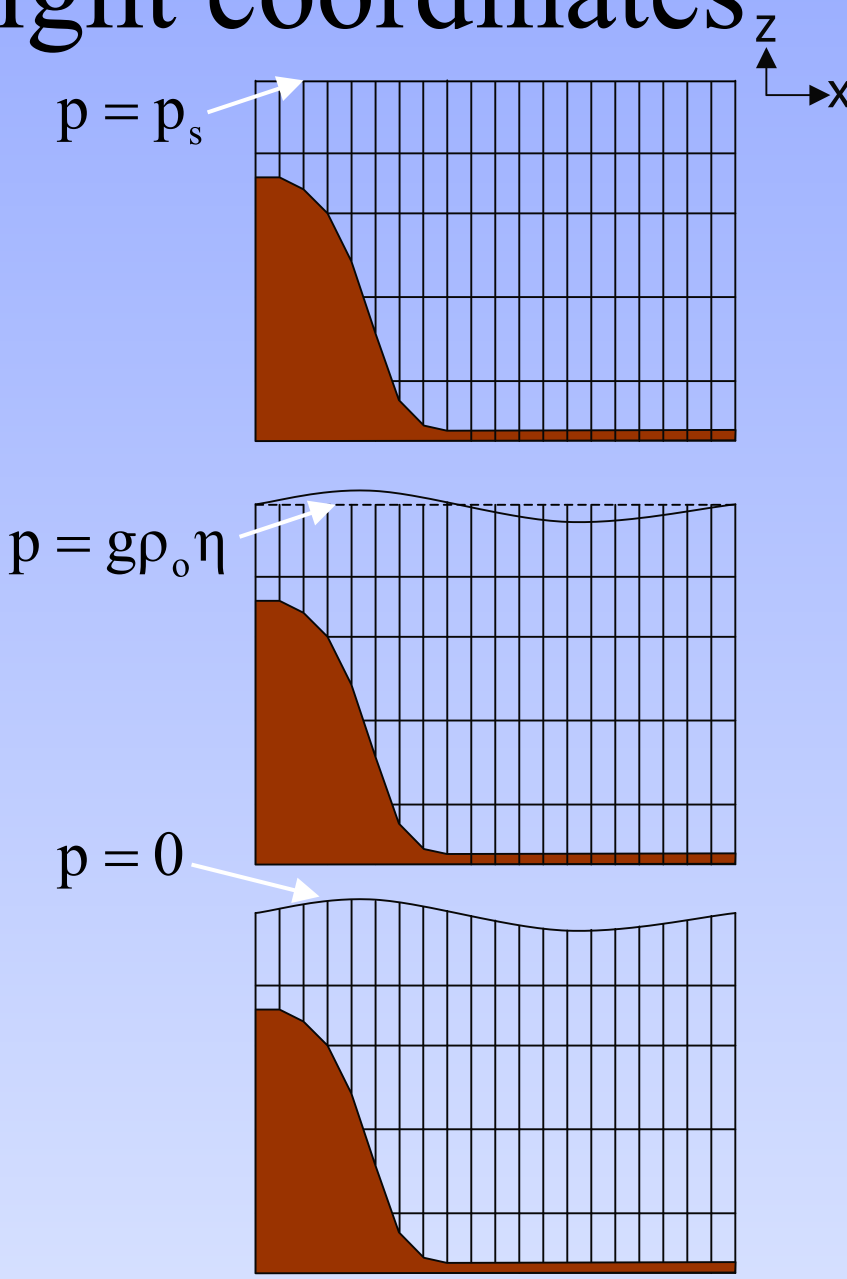
- Linear free surface
  - Semi-implicit
  - Helmholtz eq$^{\underline{n}}$ for η

$$\partial_t \eta + \nabla \cdot H\vec{v} = P - E$$

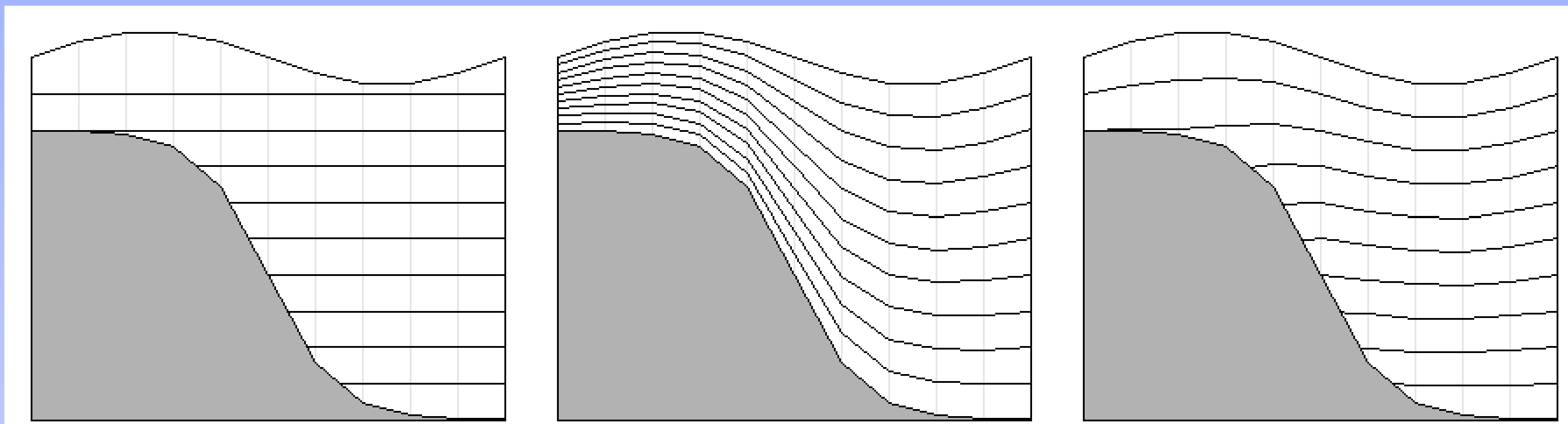
- Non-linear free surface
  - FV method for top layer

(in height coordinates)

$$\partial_t \eta + \nabla \cdot (H + \eta)\vec{v} = P - E$$



Motivation for z* coordinate



Free surface height (z) coordinate models | Terrain following coordinate (σ) models | z* coordinate

Free surface height (z) coordinate models	Terrain following coordinate (σ) models	z* coordinate
•Accurate FV topography	•Smooth topography(?)	•Best of both worlds?
•No pressure gradient errors	•Pressure gradient errors	•Irregular comp. domain
•Irreg./variab. comp. domain	•Regular comp. domain	•Fixed comp. domain
•Vanishing surface layer	•Fixed comp. domain	•Accurate external mode
	•Accurate external mode	

Stacey's z* coordinate

- Vertical motion due to external mode is absorbed into coord. system
  - more stable
  - reduced spurious fluxes associated with vert. motion
- Easier conservation than varying top layer
- There is a pressure gradient error
  - **BUT** it is small!

$$\boxed{|\nabla \eta| << |\nabla H|}$$

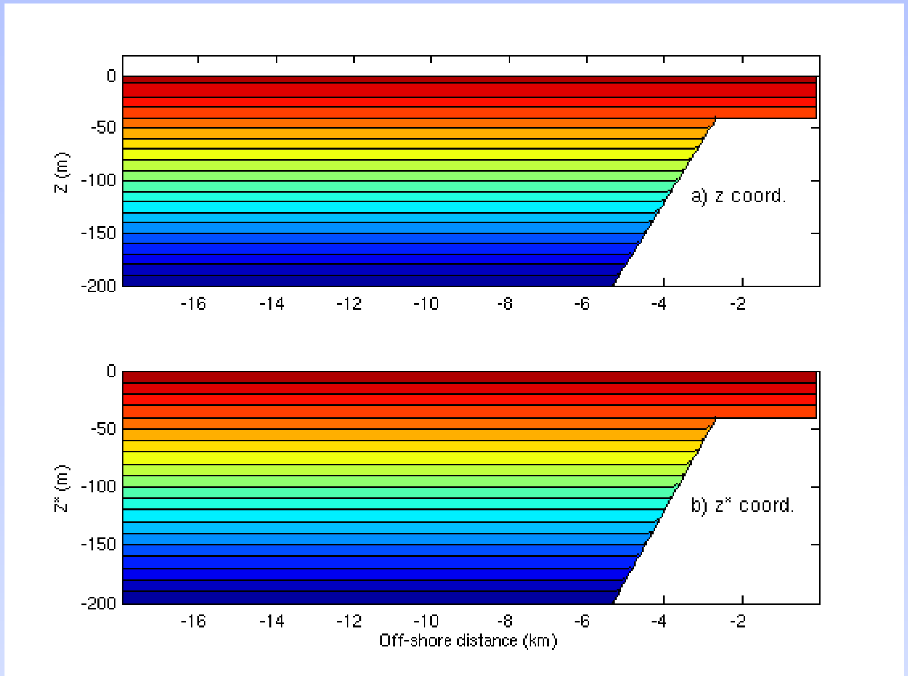
$$\boxed{z^* = \sigma H = \frac{z-\eta}{H+\eta} H}$$

$$\boxed{\partial_{z^*} z = \frac{H+\eta}{H} \sim 1}$$

Small differences from height

Internal Wave Generation

- Stratified fluid
- Barotropic forcing
- $NH = 20$ cm/s
- $U_{baro} = \pm 10$ cm/s



Hydrostatic!

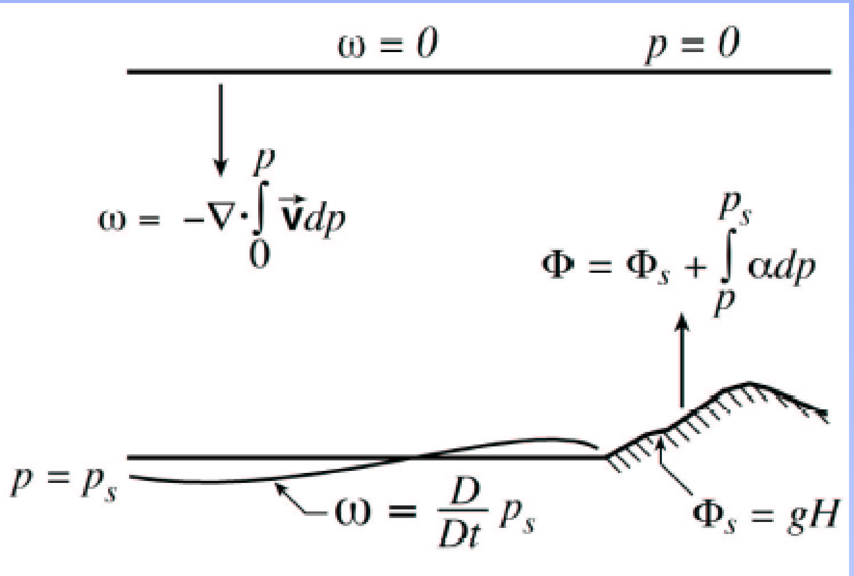
The z-p Isomorphism

- Atmospheric equations

$$D_t\vec{v}_h + 2\Omega\times\vec{v}_h + \nabla_p\Phi = \vec{F}$$
$$\alpha + \partial_p\Phi = 0$$
$$\nabla_p\cdot\vec{v}_h + \partial_p\omega = 0$$
$$\partial_t p_s + \nabla\cdot p_s\langle\vec{v}_h\rangle = 0$$

  - non-Boussinesq

$$D_t\theta = Q_\theta$$
$$\alpha = \theta\partial_p\Pi$$

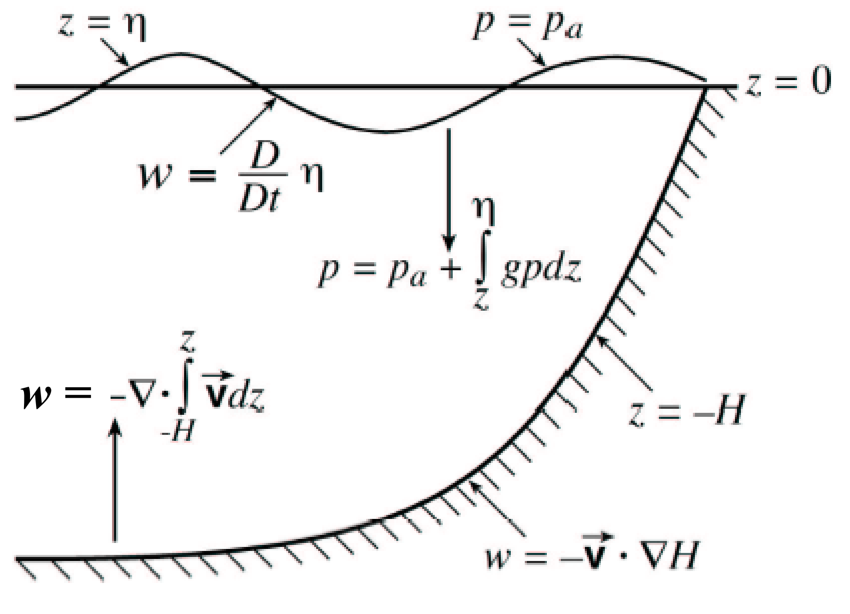


- Oceanic equations

$$D_t\vec{v}_h + 2\Omega\times\vec{v}_h + \tfrac{1}{\rho_o}\nabla_z p = \vec{F}$$
$$g\rho + \partial_z p = 0$$
$$\nabla_z\cdot\vec{v}_h + \partial_z w = 0$$
$$\partial_t\eta + \nabla\cdot(H+\eta)\langle\vec{v}_h\rangle = P - E$$

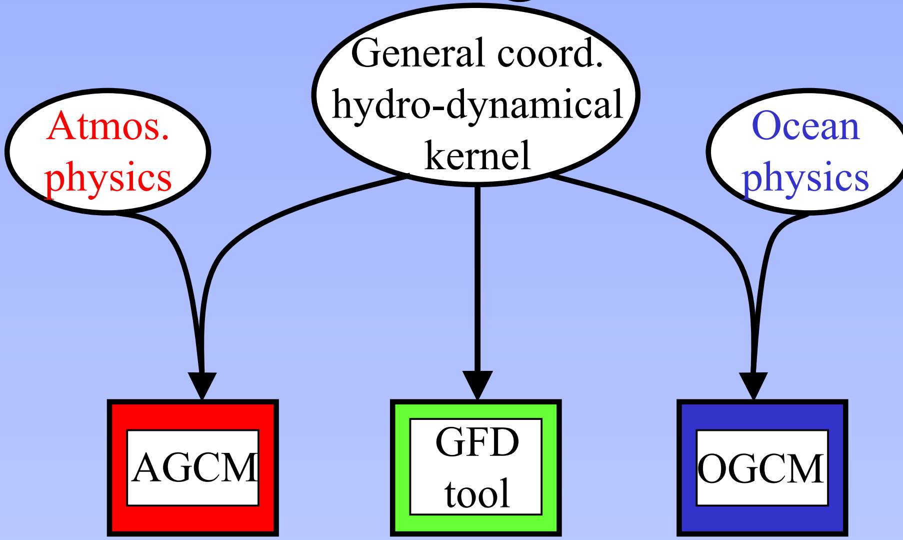
  - Boussinesq

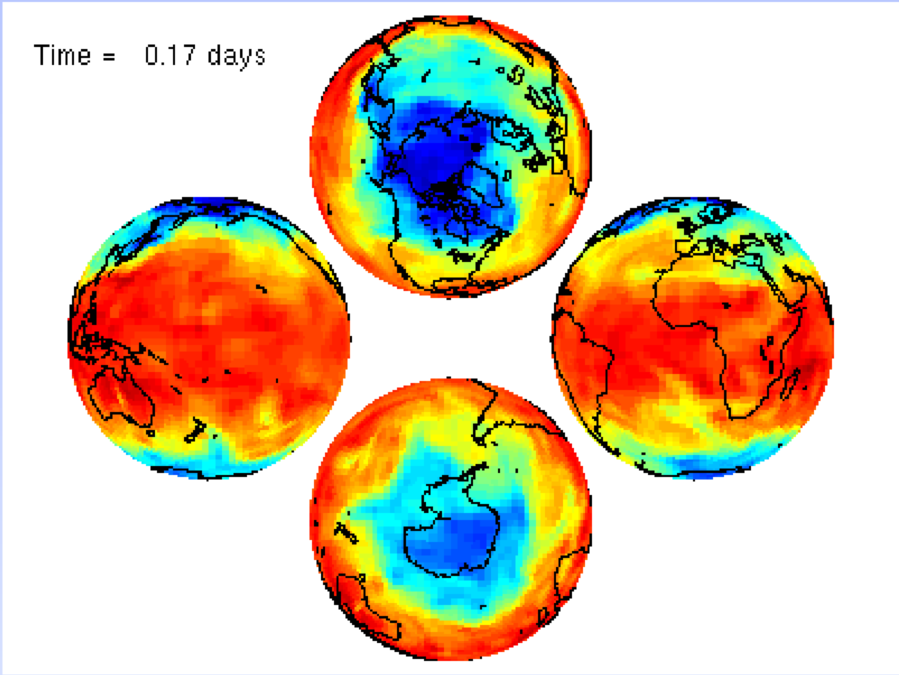
$$D_t\theta = Q_\theta$$
$$\rho = \rho(s,\theta,p)$$



Unified approach to Ocean/Atmosphere modeling

- The z-p isomorphism
  - allows same dynamical core to model either ocean or atmosphere
- Discriminate between fluids with “plug-in” physics
- Leverage developments
  - Developments for one application immediately available in the other
    - e.g. “cubed sphere”, vertical coordinate, finite volume method, etc…





The z*-p* Isomorphism

(Ocean-Atmosphere Isomorphism)

- Oceanic equations
  - Boussinesq
  - incompressible

$$D^*_t \vec{v}_h + 2\Omega\times\vec{v}_h + \frac{\rho}{\rho_o}\nabla_{z^*}\Phi + \frac{1}{\rho_o}\nabla_{z^*}p = \vec{F}$$

$$z_r g\rho + \partial_{z^*}p = 0$$

$$\nabla_{z^*}\cdot(z_r\vec{v}_h) + \partial_{z^*}(z_r w^*) = -\partial_t z_r$$

$$\partial_t\eta + \nabla\cdot(H+\eta)\langle\vec{v}_h\rangle = P - E$$

$$\partial_t(z_r\theta) + \nabla_{z^*}\cdot(z_r\vec{v}_h\theta) + \partial_{z^*}(z_r w^*\theta) = Q_\theta$$

$$z_r = \frac{H+\eta}{H}$$

- Atmospheric equations
  - non-Boussinesq
  - compressible

$$D_t\vec{v}_h + 2\Omega\times\vec{v}_h + \alpha\nabla_{p^*}p + \nabla_{p^*}\Phi = \vec{F}$$

$$\alpha + p_r\partial_p\Phi = 0$$

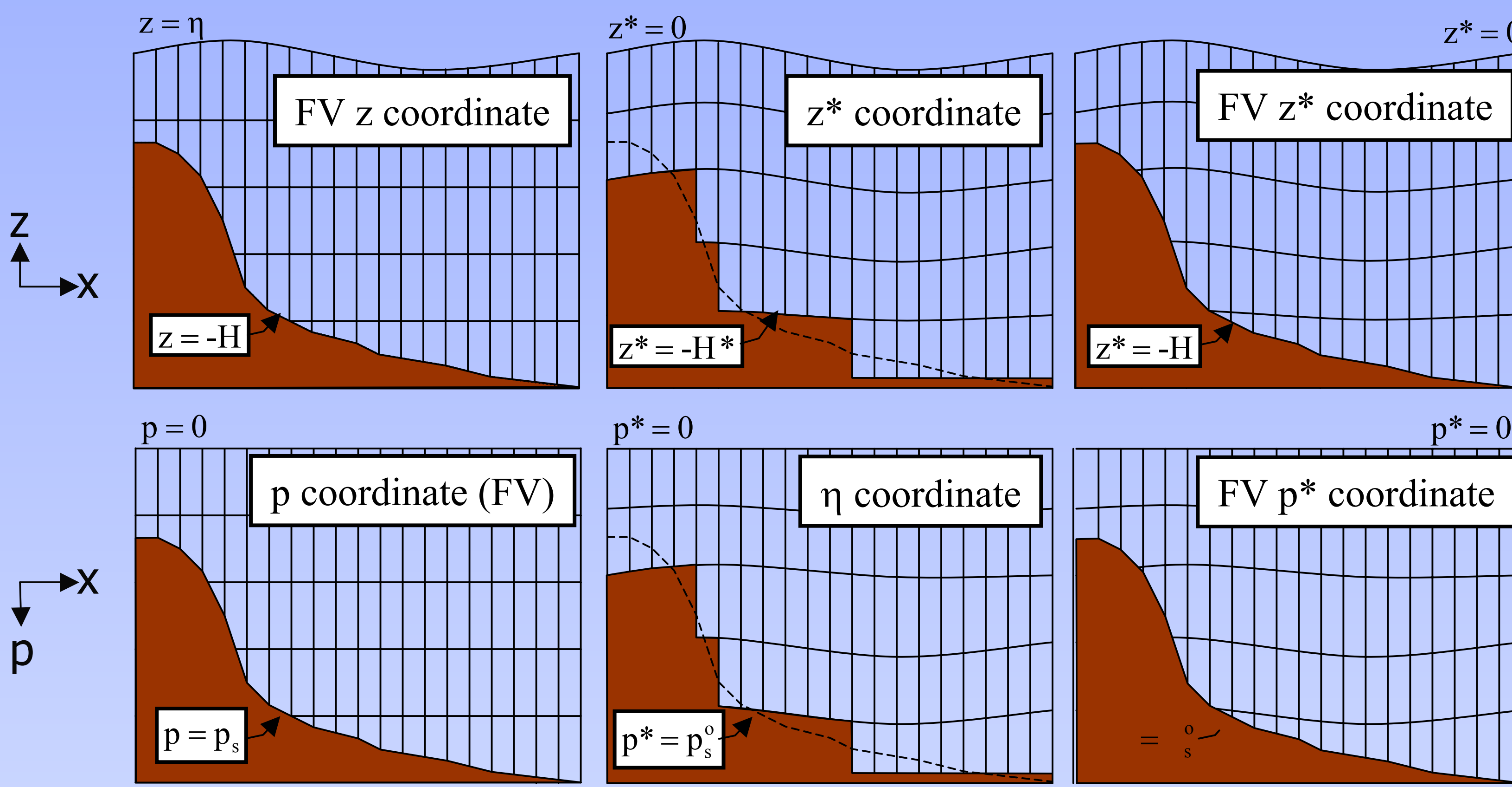
$$\nabla_{p^*}\cdot(p_r\vec{v}_h) + \partial_{p^*}(p_r\omega^*) = -\partial_t p_r$$

$$\partial_t p_s + \nabla\cdot p_s\langle\vec{v}_h\rangle = 0$$

$$\partial_t(p_r\theta) + \nabla_{p^*}\cdot(p_r\vec{v}_h\theta) + \partial_{p^*}(p_r\omega^*\theta) = Q_\theta$$

$$p_r = \frac{p_t + p_s}{p_s^o}$$

Menagerie of coordinates



The "eta" coordinate

z*-p* isomorphism

$$\boxed{z^* = \frac{z - \eta(x,y,t)}{H(x,y) + \eta(x,y,t)} H(x,y)}$$

$$\boxed{p^* = \frac{p - p^t}{p_s(x,y,t) - p^t} p_s^o(x,y)}$$

- p* is dimensional form of "eta" coordinate (units of pressure)
- Otherwise known as the "step mountain" coordinate

(Mesinger et al., MWR `88)

  - *Finite volume method avoids "step" orography*
- Presented as a fix for the PG error in terrain following coordinates
  - *"eta" looks nothing like σ coordinate*
  - *"eta"* ***is*** *a pressure-like coordinate*
- That the orography is where the "free-surface" is confusing
  - the isomorphism reveals that the B.C.'s need not be confused

Method Of Lines/Direct Space Time

Hunsdorfer, 1995

- Method Of Lines (MOL)
  - Space and time considered separately
  - Convergence of model limited by lowest order scheme if Δt and Δx are related (e.g. near CFL limit)
  - Explicit high order time discretization requires
    - Either more time levels (e.g. AB3) costly
    - Or more stages (e.g. RK4) costly
- Direct Space Time (DST)
  - Discretize all dimensions together
  - Can find stable two time level, single stage (like "forward") schemes with same spatial stencil efficient
    - Well known example: Lax-Wendroff !

Direct Space Time

Hunsdorfer & Trompert, 1995

- Consider flux form of $\partial_t\theta + u\partial_x\theta$ (u is constant, >0)
- Result of DST looks like forward method

$$\frac{1}{\Delta t}\left(\theta_i^{n+1} - \theta_i^n\right) + \frac{1}{\Delta x}\left(F_{i+½}^{US} - F_{i-½}^{US}\right) = \partial_t\theta + u\partial_x\theta \quad + O\left(\Delta t, \Delta x\right)$$

$$\frac{1}{\Delta t}\left(\theta_i^{n+1} - \theta_i^n\right) + \frac{1}{\Delta x}\left(F_{i+½}^{LW} - F_{i-½}^{LW}\right) = \partial_t\theta + u\partial_x\theta \quad + O\left(\Delta t^2, \Delta x^2\right)$$

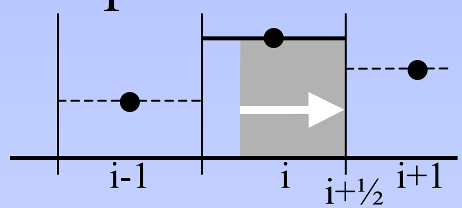
$$\frac{1}{\Delta t}\left(\theta_i^{n+1} - \theta_i^n\right) + \frac{1}{\Delta x}\left(F_{i+½}^{DST3} - F_{i-½}^{DST3}\right) = \partial_t\theta + u\partial_x\theta \quad + O\left(\Delta t^3, \Delta x^3\right)$$

- Fluxes modified to balance time/space truncation errors

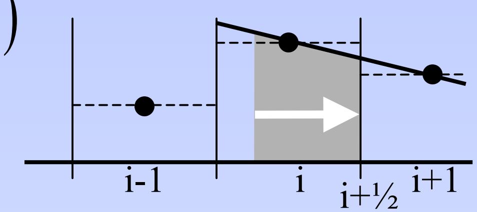
$$F_{i+½}^{US} = u\theta_i^n$$

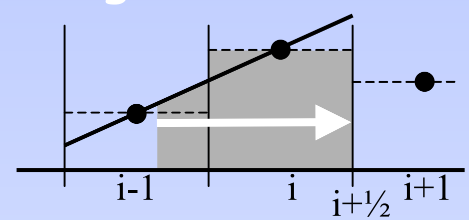
$$F_{i+½}^{LW} = F_{i+½}^{US} + \tfrac{1}{2}u(1-C)\left(\theta_{i+1}^n - \theta_i^n\right)$$

$$F_{i+½}^{DST3} = F_{i+½}^{LW} - \tfrac{1}{6}u(1-C^2)\left(\theta_{i+1}^n - 2\theta_i^n + \theta_{i-1}^n\right)$$

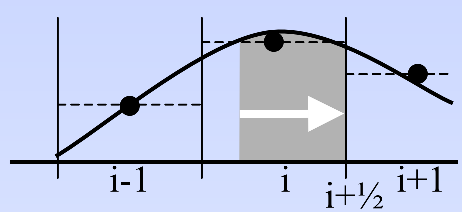


Hunsdorfer & Spee, MWR 1995
"Integer advection"

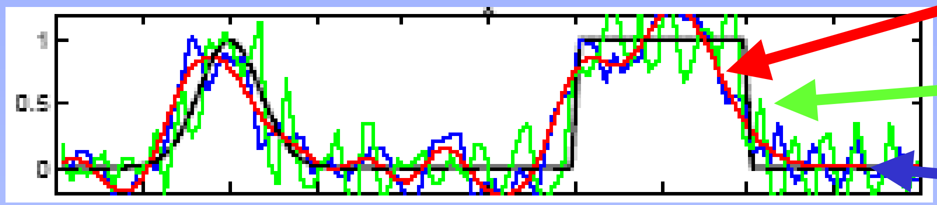




- Simple. Elegant.
- …add integer advection
- Beautiful!



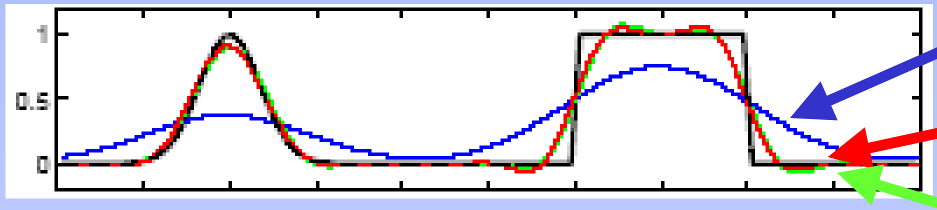
Advection schemes



Law-Wendroff (Forward)

Centered 2nd order +AB3

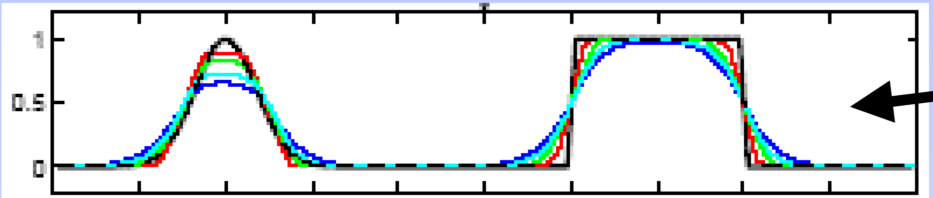
Centered 4th order +RK4



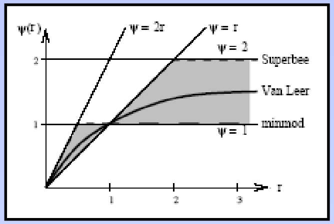
Upwind 1st order (Forward)

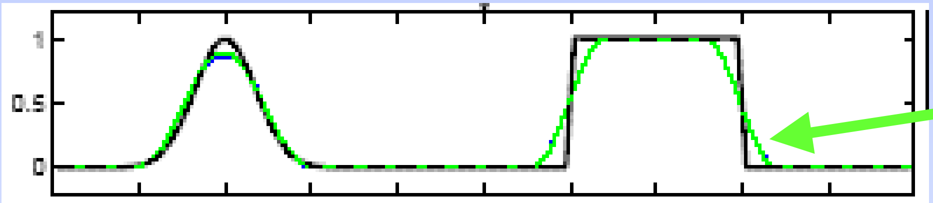
Upwind 3rd order +AB3

Direct 3rd order (Forward)



Flux-limited 2nd order
- van Leer, Superbee





Flux limited direct 3rd order (Sweby)

Accuracy *versus* **Fidelity** ?

Multi-dimensional advection

- Multi-dimensional flow appears divergent in any one direction
- 1-D properties of schemes need non-divergent flow

$\approx u\partial_x\theta$

$$\theta^{n+1/3} = \theta^n - \Delta t\left[\partial_x\left(u\theta^n\right) - \theta^n\partial_x u\right]$$

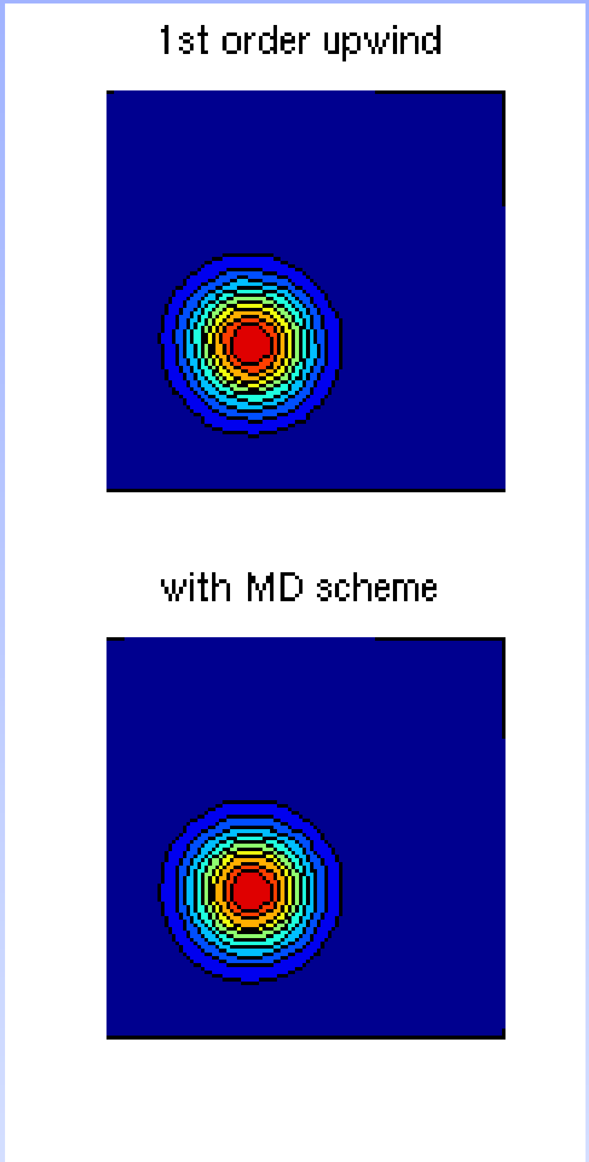
$$\theta^{n+2/3} = \theta^{n+1/3} - \Delta t\left[\partial_y\left(v\theta^{n+1/3}\right) - \theta^n\partial_y v\right]$$

$$\theta^{n+3/3} = \theta^{n+2/3} - \Delta t\left[\partial_z\left(w\theta^{n+2/3}\right) - \theta^n\partial_z w\right]$$

$$\theta^{n+1} = \theta^{n+3/3} - \Delta t\theta^n\left(\partial_x u - \partial_y v - \partial_z w\right)$$

=0

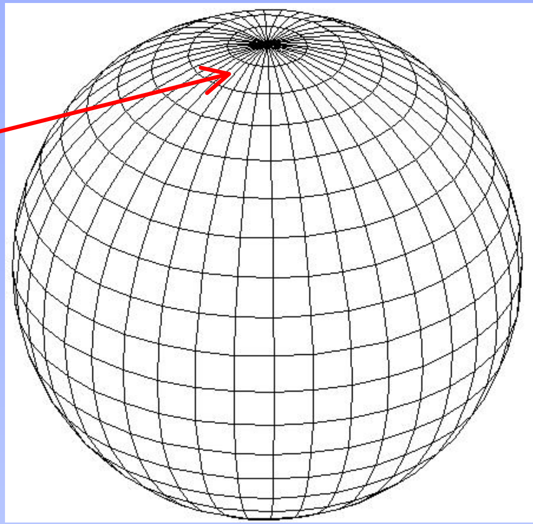
- Extends properties from 1-D to N-D
- Splitting error is minimized by changing order of directions for each consecutive time-step
- Same stability as if 1-D



Gridding the sphere

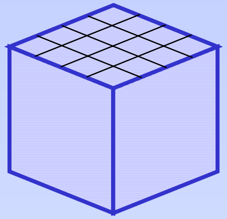
- Latitude-longitude grid
  - converging meridians
  - prohibitive scaling

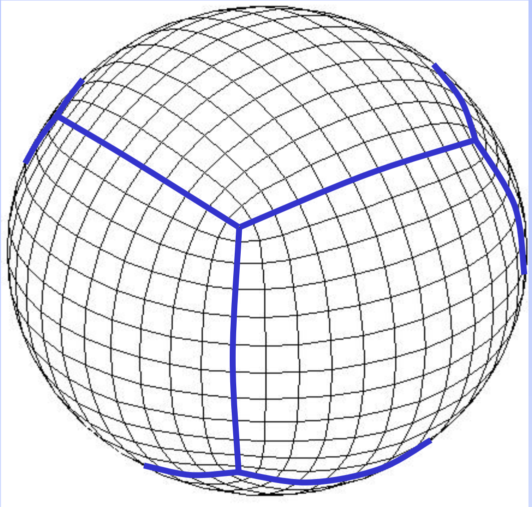
$$\boxed{\Delta x_{min} \sim N^{-2}}$$



- Expanded spherical cube
  - conformal mapping
  - near uniform resolution
  - much improved scaling

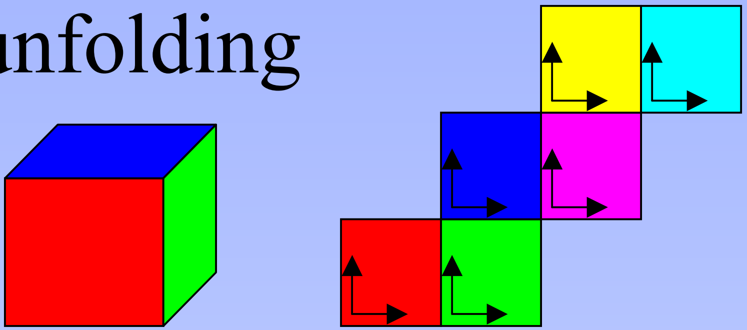
$$\boxed{\Delta x_{min} \sim N^{-4/3}}$$



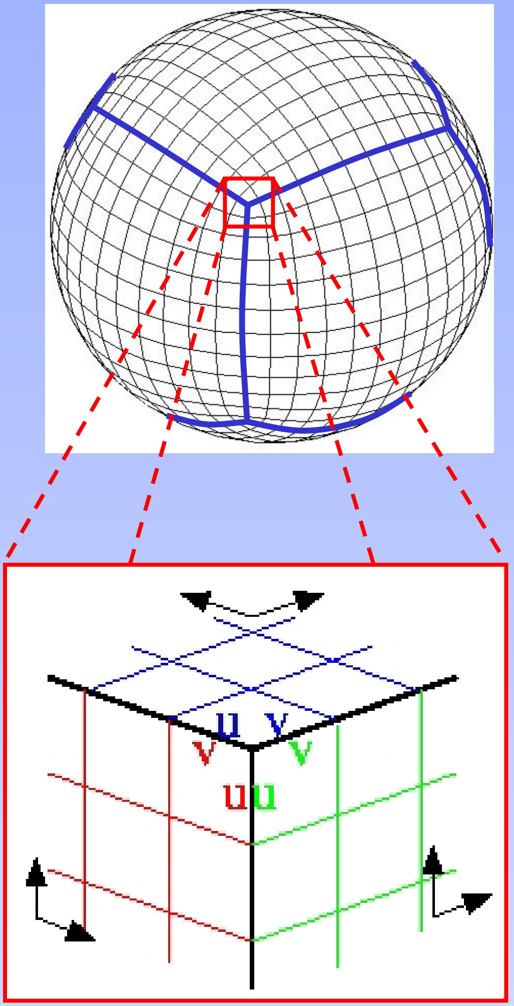


Gridding the sphere

- Map the faces of a cube into memory by unfolding



- Corners of cube ⇒ 8 singularities
  - need components of vector quantities on singularities
  - degeneracy of coordinate systems



$$\zeta = \frac{1}{e_1 e_2}\left(\frac{\partial}{\partial x_1} e_2 v - \frac{\partial}{\partial x_2} e_1 u\right)$$

"Finite Volume" v's tensorial formalism

$$\boxed{\zeta = \frac{1}{e_1 e_2}\left(\frac{\partial}{\partial x_1} e_2 v - \frac{\partial}{\partial x_2} e_1 u\right)}$$

- Gradients across corners occur in finite difference mindset
- Integral formulation avoids any ambiguity about discretization

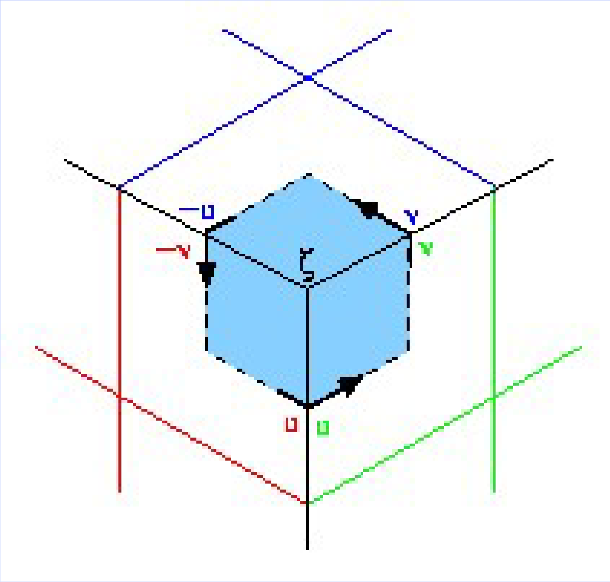
e.g. e_1=r cosφ, e_2=r, x_1=λ, x_2=φ

$$\zeta = \frac{1}{r^2 \cos\varphi\, \Delta\lambda\, \Delta\varphi}\left(\Delta\varphi\, \delta_i \mathrm{r}\, v - \Delta\lambda\, \delta_j \mathrm{r} \cos\varphi\, u\right)$$

$$\boxed{\zeta = \frac{\Gamma}{A} = \frac{1}{A}\left(\delta_i \Delta y\, v - \delta_j \Delta x\, u\right)}$$

e.g. Δx= Δλ r cosφ, Δy=r Δφ, A=∫ Δx r dφ

$$\zeta = \frac{1}{r^2 \Delta\lambda\,(\sin\varphi_2 - \sin\varphi_1)}\left(\delta_i \Delta\varphi\, \mathrm{r}\, v - \delta_j \Delta\lambda\, \mathrm{r} \cos\varphi\, u\right)$$



Vector Invariant Eq^{ns}

- Tensorial form of conservative SWEs

$$\partial_t u + \frac{u}{e_1}\partial_{x_1} u + \frac{v}{e_2}\partial_{x_2} u - \left[f + \frac{v\partial_{x_1} e_2 - u\partial_{x_2} e_1}{e_1 e_2} \right] v + \frac{1}{e_1}\partial_{x_1} gh = F_u$$

$$\partial_t v + \frac{u}{e_1}\partial_{x_1} v + \frac{v}{e_2}\partial_{x_2} v + \left[f + \frac{v\partial_{x_1} e_2 - u\partial_{x_2} e_1}{e_1 e_2} \right] u + \frac{1}{e_2}\partial_{x_2} gh = F_v$$

$$\partial_t h + \frac{1}{e_1 e_2}\partial_x e_2 hu + \partial_y e_1 hv = 0$$

- Finite volume method applied to Vector Invariant SWEs
  - described entirely in terms of lengths and areas
  - no “metric terms”

$$\partial_t u - (f+\zeta)v + \frac{1}{\Delta x}\delta_i\left(gh + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2\right) = F_u$$

$$\partial_t v + (f+\zeta)u + \frac{1}{\Delta y}\delta_j\left(gh + \tfrac{1}{2}u^2 + \tfrac{1}{2}v^2\right) = F_v$$

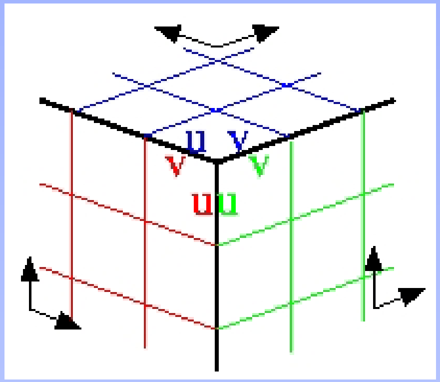
$$\partial_t Ah + \delta_i hu\Delta x + \delta_j hv\Delta y = 0$$

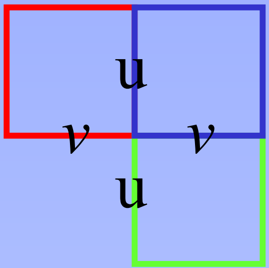
$$\zeta = \frac{\Gamma}{A} = \frac{1}{A}\left(\delta_\mathrm{i} v\Delta y - \delta_\mathrm{j} u\Delta x\right)$$

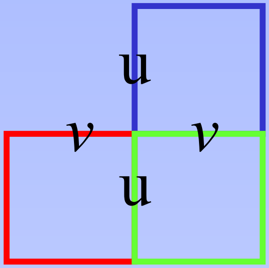
Non-commuting operators

- Normally we can assume:

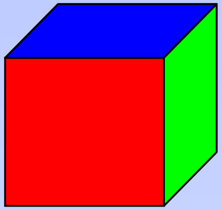
$$\overline{\overline{v}^{j}}^{i} = \overline{\overline{v}^{i}}^{j}$$







  - near singularities this fails
  - the sequence of operations matters
- Working rule: interpolate via cell centers:



$$\partial_t u - \overline{f+\zeta}^{j}\, \overline{\overline{v}^{j}}^{i} + \cdots$$

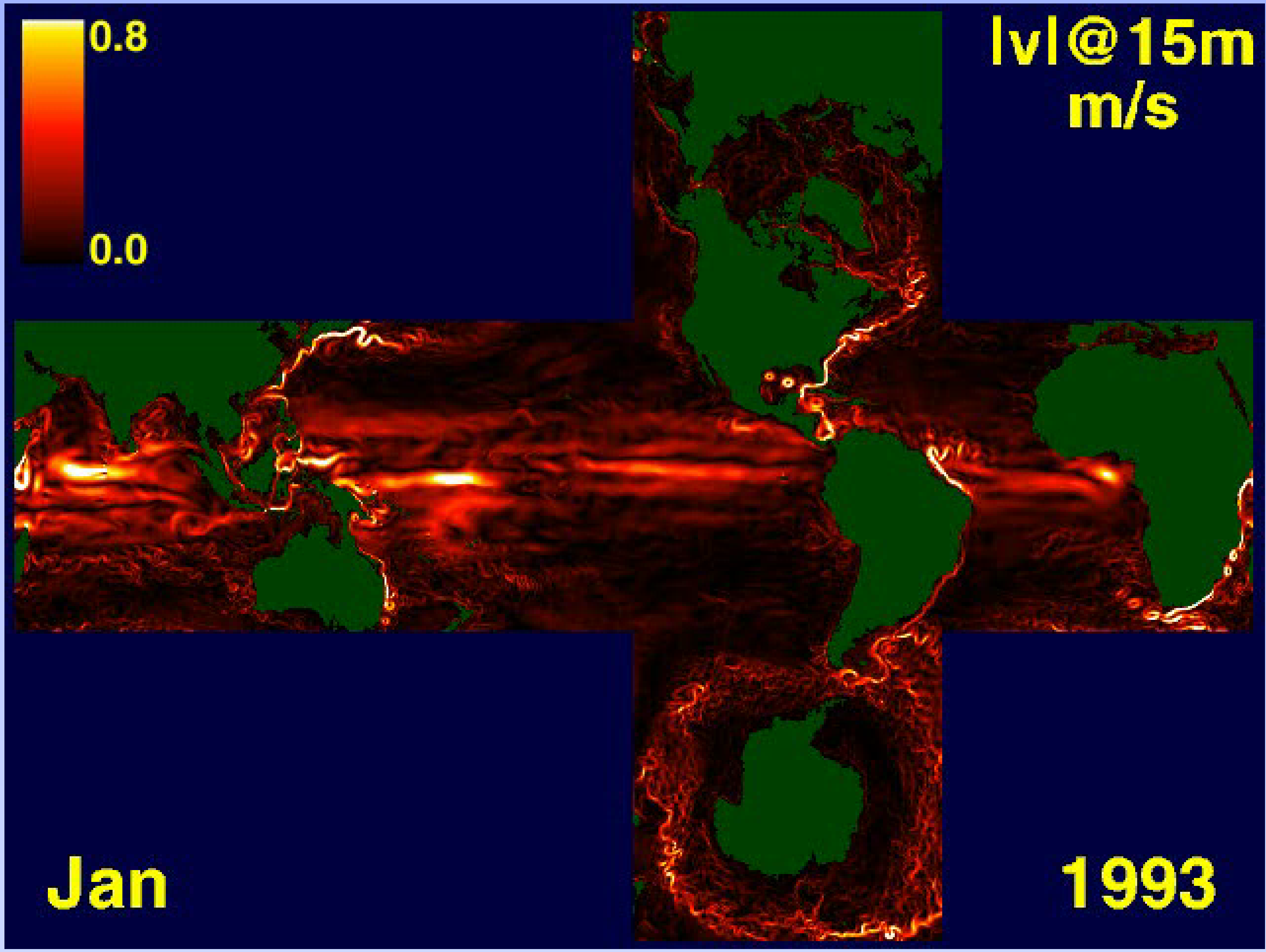
$$\partial_t v + \overline{f+\zeta}^{i}\, \overline{\overline{u}^{i}}^{j} + \cdots$$

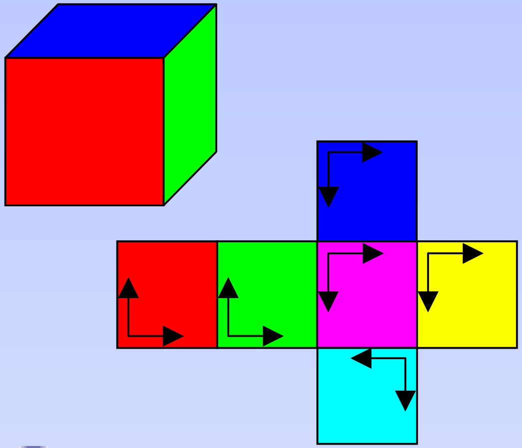
  - conserves something similar to enstrophy…

Eddy-permitting global ocean

ECCO project D. Menemenlis, JPL/NASA

- $C_{512} = 512^2 \cdot 6$
- $7 \text{ km} \leq \Delta x \leq 19 \text{ km}$
- 10 years/day
- Includes Arctic
  - has sea-ice





• 480 SGI Altix processors, NASA

Issues and future

- Planning a 2-5 km cubed run
  - Better grid generation?
  - Non-orthogonal coordinate systems?
  - Qualitative changes in solution
  - Eddy statistics/state estimation (Ferreira & Marshall) ECCO
- Hybrid coordinates and z**
  - spurious diabatic fluxes are a community wide concern
- Flux limiters on vorticity flux
  - needs similar treatment for K.E. term/eqn?
- Direct method for momentum eqns? Burgers equation
  - currently using 3-4 time levels
  - 2 level scheme would shrink the state vector
    - adjoint, D.A., restarts, performance, …