Flow-dependent transforms

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1 Balanced Dynamics and VAR

• Preconditioning for VAR assimilation
• Need 'nearly' uncorrelated variables: Met Office scheme

$$\{u, v, w, \theta, \rho, p, q\} \quad \underbrace{\leftrightarrow}_{\mathbf{T}_p,\, \mathbf{U}_p \ -\text{transforms}} \quad \{\psi, \chi, {}^{A}P, \mu\} \tag{1}$$

${}^{A}P$ is an unbalanced pressure, ψ and χ are the stream function and velocity potential for the 'horizontal' wind; μ is a moisture (humidity) variable.
• The four variables are required to initialize a hydrostatic primitive equation model; w, θ, ρ are diagnosed.

1.1 $\mathbf{T}_p$-transform

If $\mathbf{u} = (u, v)$, $\nabla_h = (\partial_x, \partial_y)$, then currently

$$\nabla_h \times \mathbf{u} = \nabla_h^2 \psi, \ \nabla_h . \mathbf{u} = \nabla_h^2 \chi, \tag{2}$$

and

$${}^{A}P = {}^{H}P - {}^{F}P \tag{3}$$

where ${}^{H}P$ is a hydrostatically balanced pressure and ${}^{F}P$ is a geostrophically balanced pressure obtained from linear balance equation

$$\nabla_h . (f \mathbf{k} \times \mathbf{u}) + \nabla_h . \left(\frac{1}{\rho} \nabla_h {}^{G}P \right) = 0.$$

1.2 $\mathbf{U}_p$-transform

Inverse of (2)

$$\mathbf{u} = \nabla_h \chi + \mathbf{k} \times \nabla_h \psi \tag{4}$$

and once the horizontal wind field is determined we use linear balance to obtain ${}^{G}P$ and then

$${}^{H}P = {}^{A}P + {}^{G}P$$

etc.
The increment to vertical velocity is calculated by either integrating a continuity equation (several choices here), or Richardson's equation, or set equal to zero(!)

2 Use of Potential Vorticity in VAR

(Wlasak, Cullen & Bannister)

The nonlinear shallow water equations take the form:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + f\mathbf{k}\times\mathbf{v} + g\nabla h = 0 \tag{5}$$

$$\frac{\partial h}{\partial t} + \nabla\cdot(h\mathbf{v}) = 0 \tag{6}$$

where h the height of the fluid, and $\mathbf{v}$ is the horizontal vector wind. The acceleration due to gravity, g, is considered to be a constant and the Coriolis parameter, f, on the sphere is a function of latitude only.

• For the most part in mid-latitudes, there is a separation in time scales between the slow Rossby modes and fast inertio-gravity modes. As stated earlier, we are interested in choosing variables that demonstrate this dynamical separation.

One of the key non-dimensional constants is the Burger number, B_u, which is defined as the ratio of the Rossby radius of deformation, L_R, and the horizontal characteristic length scale L:

$$\begin{aligned} B_u &= \frac{L_R}{L}, \\ L_R &= \frac{\sqrt{gH}}{f}, \end{aligned} \tag{7}$$

where f is a typical value for the Coriolis parameter and H is the characteristic depth of the fluid. The Rossby number is defined in the usual way

$$R_o = \frac{U}{fL}, \tag{8}$$

where U is the characteristic velocity.

For the shallow water equations, in the asymptotic limit as the Burger number gets large, $B_u >> 1$, where the Rossby number is kept small, $R_o << 1$, the typical spatial differences in depth become increasingly less important compared to the effect of the characteristic depth and in this case, the continuity equation effectively degenerates into a 2D incompressibility condition. This enforces a *non-divergent flow described by incompressible 2D Euler* as

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} + f\mathbf{k}\times\mathbf{v} + g\nabla h = 0, \tag{9}$$

$$\nabla\cdot\mathbf{v} = 0. \tag{10}$$

Taking the vertical component of the curl of the momentum equation (9) gives the barotropic vorticity equation

$$\frac{\partial \nabla^2\psi}{\partial t} + \mathbf{v}_\psi\cdot\nabla\left(f + \nabla^2\psi\right) = 0, \tag{11}$$

where $\mathbf{v}_\psi$ is the rotational, non-divergent part of the wind and the stream function ψ is defined by

$$\nabla^2\psi = \mathbf{k}\cdot\nabla\times\mathbf{v}. \tag{12}$$

1. As (11) is a single equation with a single time derivative, it has just one eigenmode which approximates the slow mode of the shallow water equations as the Burger number becomes large.

2. The barotropic vorticity equation is an example of a balanced model, as it is a reduced model that approximates the shallow water equations in the asymptotic limit $R_o << 1$, $B_u >> 1$.

3. From this equation we see that in this limit the streamfunction is the appropriate variable to approximate the slow dynamics (cf. Met O scheme)

2.1 Linear balance equation

A consequence of inviscid incompressible 2D Euler is that if we take the divergence of the momentum equation (9), we are left with the Charney-Bolin balance equation

$$g\nabla^2 h = \nabla \cdot f\nabla\psi - 2(\psi_{xy}^2 - \psi_{xx}\psi_{yy}). \tag{13}$$

Applying scale analysis to this equation shows that for $R_o << 1$ two terms are typically a factor of ten larger than the rest. Taking these two terms on their own gives the *linear balance equation* as

$$g\nabla^2 h = \nabla \cdot f\nabla\psi. \tag{14}$$

2.2 The 'LB method'

In data assimilation we are interested in an incremental linearised formulation. In particular, the height and wind increments are defined by

$$\begin{aligned} h' &= h - \overline{h}, \\ \mathbf{v}' &= \mathbf{v} - \overline{\mathbf{v}}. \end{aligned}$$

The linearisation states $\overline{h}, \overline{u}, \overline{v}$ are assumed to be a function of latitude only.

The streamfunction increment can be considered to represent the balanced part of the incremental flow and the linear balance equation (14) can be used to find the balanced height increment h'_b from the streamfunction increment ψ'.

• This method for decomposing the flow is referred to here as the LB method in which three control variables (ψ', h'_{res}, χ'), one balanced and two unbalanced, are produced by:

1. assuming that the rotational wind describes the slow dynamics

2. representing the rotational wind through the calculation of the streamfunction increment from the wind increments using equation (12) and assuming it is balanced, i.e. $\psi'_b = \psi'$;

3. using the linear balance equation (14) to derive the associated balanced height increment h'_b and the unbalanced height increment $h'_{ub} = h' - h'_b$.;

4. calculating the remaining part of the flow, the divergent wind, described by velocity potential increment χ' and given by
$$\nabla^2\chi' = \nabla \cdot \mathbf{v}'. \tag{15}$$

2.3 PV control variable

Potential vorticity, q, is considered to be a key dynamical quantity that can capture atmospheric flow features such as frontogenesis, cyclogenesis and general circulation.

In the context of the *shallow water equations*, its form is a generalisation of 2D Euler and is given by

$$q = \frac{f + \nabla^2 \psi}{h}, \tag{16}$$

with the property that it is conserved.

The linearised potential vorticity increment q' is defined with respect to a reference state $\overline{q}$ that satisfies the nonlinear potential vorticity equation

$$\overline{q} = \frac{f + \nabla^2 \overline{\psi}}{\overline{h}}, \tag{17}$$

with $\nabla^2 \overline{\psi}$ and $\overline{h}$ being reference states for the relative vorticity and height.

Linearising the PV equation (16) about $\overline{q}$ then defines the potential vorticity increment q' as

$$\frac{q'}{\overline{q}} = \frac{\nabla^2 \psi'}{f + \nabla^2 \overline{\psi}} - \frac{h'}{\overline{h}}. \tag{18}$$

2.4 The PV method

The following coupled system, derived from (18) and (14), defines our decomposition of the slow dynamics:

$$\nabla \cdot f \nabla \psi'_b - g \nabla^2 h'_b = 0, \tag{19}$$

$$\nabla^2 \psi'_b - \overline{q} h'_b = \overline{h} q', \tag{20}$$

which we solve, simultaneously for ψ'_b and h'_b, where $\overline{q}$, $\overline{h}$ and q' are known. The term $\overline{h}q'$ is precalculated from ψ' and h' by just rearranging equation (18) as

$$\overline{h} q' = -\overline{q} h' + \nabla^2 \psi'. \tag{21}$$

In addition, $\nabla^2 \psi'$ is obtained from the full wind increments $\mathbf{v}'$ using the equation $\nabla^2 \psi' = \mathbf{k} \cdot (\nabla \times \mathbf{v}')$.

The unbalanced rotational wind can be obtained by subtracting the balanced wind and height from the full rotational wind and height,

$$\psi'_{ub} = \psi' - \psi'_b, \tag{22}$$

$$h'_{ub} = h' - h_b \tag{23}$$

where the unbalanced rotational wind is defined to be $\mathbf{v}'_{ub} = \mathbf{k} \times \nabla \psi'_{ub}$. The unbalanced height is denoted by h'_{ub} and again, on the right hand side, we use known full increments ψ' and h'.

The PV method is defined by calculating the following three control variables: balanced streamfunction increment ψ'_b, an unbalanced height increment h'_{ub} and velocity potential increment χ'.

The method consists of the following steps:

1. assume that, instead of conserving ψ' as in the LB method, the increment $\overline{h}q'$ is conserved

2. solve the linear balance and linearised PV equations (19), (20) simultaneously to give balanced stream-function increments ψ'_b and balanced height increments h'_b;

3. derive the unbalanced height increment h'_{ub} by subtracting the balanced height increment h'_b from the full increment h' as in equation (23);

4. calculate the velocity potential increment, χ', by calculating the divergence of the wind and inverting a Laplacian as in equation (15).

2.5 Dependence of linearised PV on Burger number

Assume constant Coriolis parameter.

Key relationships are

$$\frac{q'}{\overline{q}} = -N\frac{h'}{\overline{h}}$$

$$\left(1-\frac{1}{N}\right)\frac{q'}{\overline{q}} = \frac{\nabla^2\psi'}{\nabla^2\overline{\psi}+f_0}$$

with

$$N = 1+\frac{f_0 B_u^2}{\nabla^2\overline{\psi}+f_0}.$$

As $B_u > 0$, $N > 1$.

1. So large B_u means PV dominated by vorticity

2. Low B_u means PV dominated by height

Similarly, if

$$P = 1+\frac{\nabla^2\overline{\psi}+f_0}{f_0 B_u^2}$$

then

$$\left(1-\frac{1}{P}\right)\frac{q'}{\overline{q}} = -\frac{h'}{\overline{h}}$$

and

$$\frac{q'}{\overline{q}} = P\frac{\nabla^2\psi'}{\nabla^2\overline{\psi}+f_0}.$$

3 Using ideas from Hamiltonian mechanics

(Fletcher)

At present, horizontal momentum balance is defined by the Helmholtz decomposition into rotational and irrotational flow:

$$\mathbf{u} = \underbrace{\mathbf{k}\times\nabla\psi}_{\text{balanced}} + \underbrace{\nabla\chi}_{\text{unbalanced}} . \tag{24}$$

The problem with this (purely kinematic) definition is that *all* the divergent flow is treated as unbalanced.

To improve on this decomposition we propose to use, in place of ψ, a scalar φ, which is obtained from the (nonlinear) Monge-Ampère equation arising in the asymptotic expansion of the (vertical component of the) vorticity to second order in the Rossby number:

$$\zeta^{\mathrm{C}} \equiv \left[f + \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} + \frac{a}{f}\frac{\partial(u_g, v_g)}{\partial(x,y)} \right], \tag{25}$$

where

$$u_g = -\frac{1}{f}\frac{\partial \varphi}{\partial y}, \; v_g = \frac{1}{f}\frac{\partial \varphi}{\partial x}, \; (\varphi = gh),$$

is the geostrophic wind, ζ^{C} is the 'constrained' vorticity — i.e. defined by some balance condition, and the constant, a, takes the value of either 0 (geostrophic balance), +1 (semi-geostrophic balance) or -2 (Charney-Bolin balance).

It can be shown (McIntyre and Roulstone 1996) that $\zeta^{\mathrm{C}} = f + \nabla_h \times \mathbf{u}^{\mathrm{C}}$, where

$$\mathbf{u}^{\mathrm{C}} = \mathbf{u}_g - \alpha \mathbf{k} \times f^{-1} \mathbf{u}_g . \nabla \mathbf{u}_g . \tag{26}$$

In (26), α is a constant, related to a in (25), which takes the value $\frac{1}{2}$ (semi-geostrophic balance), 0 (geostrophic balance) or -1 (Charney-Bolin balance). Typically $\nabla . \mathbf{u}^{\mathrm{C}} \neq 0$.

In the control variable transforms, we replace $\mathbf{k} \times \nabla \psi$ by $\mathbf{u}^{\mathrm{C}}$. Increments to a balanced mass field are obtained by linearising the potential vorticity

$$q = \frac{(f + \zeta^{\mathrm{C}})}{\varphi}$$

(Rossby's formula) using

$$\varphi^b = \overline{\Phi} + \varphi^{b\prime},$$

in (25) and solving the resulting variable coefficient (Poisson) equation for $\varphi^{b\prime}$ given $q^\prime$ from model data ($\overline{\Phi}$, $\varphi^\prime$, and $\zeta^\prime = \nabla_h \times \mathbf{u}^\prime$).

The 'unbalanced velocity potential', χ^{S}, and the 'unbalanced stream function', ψ^{S}, will be defined by (cf. the second equation in (2))

$$\nabla_h^2 \chi^{\mathrm{S}} = \nabla.(\mathbf{u} - \mathbf{u}^{\mathrm{C}}) \equiv \nabla.\mathbf{u}^{\mathrm{S}}, \;\; \nabla_h^2 \psi^{\mathrm{S}} = \nabla_h \times \mathbf{u}^{\mathrm{S}}, \tag{27}$$

where the last equality on the right defines the 'velocity-split', $\mathbf{u}^{\mathrm{S}}$.

The unbalanced mass field, $\varphi^{ub\prime}$ is defined as a solution of the linearised PV equation, when $q^\prime = 0$.

The U_p transform is defined by the following steps (defined in terms of increments)

1. From φ^b calculate $\mathbf{u}^{\mathrm{C}}$ (differentiation)
2. From ψ^{S} and χ^{S} calculate $\mathbf{u}^{\mathrm{S}}$ (*ditto*)
3. Calculate $\mathbf{u} = \mathbf{u}^{\mathrm{C}} + \mathbf{u}^{\mathrm{S}}$
4. Calculate φ^{ub} and add this to φ^b to obtain the increment to the mass field

Note that $\mathbf{u}^C$ could be used to calculate balanced increments to vertical motion through Richardson's equation (may be important/necessary in projecting information from the assimilation of moisture onto balanced dynamical modes).

• How do we cope with variable-f?

3.1 The Slow Equations (Lynch *Q.J.R. Meteorol.Soc.*, 115, 1989)

Shallow water equations

$$\zeta_t + f\delta = -N^\zeta \tag{28}$$
$$\delta_t - f\zeta + \nabla^2\Phi = -N^\delta \tag{29}$$
$$\Phi_t + \overline{\Phi}\delta = -N^\Phi \tag{30}$$

where $N^\zeta, N^\delta, N^\Phi$ represent the nonlinear terms

$$N^\zeta \equiv \mathbf{u}.\nabla\zeta + \zeta\delta + (\beta_y v + \beta_x u) \tag{31}$$
$$N^\delta \equiv \mathbf{u}.\nabla\delta + \delta^2 + (\beta_y u - \beta_x v) - 2J(u,v) \tag{32}$$
$$N^\Phi \equiv \mathbf{u}.\nabla\Phi + (\Phi - \overline{\Phi})\delta \tag{33}$$

and the subscript 't' denotes the local time derivative.

The tendencies of the divergence and geostrophic imbalance ($\varepsilon \equiv \nabla^2\Phi - f\zeta$) project onto gravity modes, and we set these tendencies equal to zero to define balance:

$$\nabla^2\Phi - f\zeta = -N^\delta \tag{34}$$
$$[\nabla^2 - f^2/\overline{\Phi}](\overline{\Phi}\delta) = -(\nabla^2 N^\Phi - f N^\zeta). \tag{35}$$

The (conserved) PV is defined by

$$q = \frac{f+\zeta}{\Phi}$$

and we use q, (34) and (35) to define balanced mass and wind fields: (Φ^b, ψ^b, χ^b).

Imbalance is defined by $q = 0$, and we use the divergence tendency, δ_t, and the tendency of the geostrophic imbalance, ε_t, to define $(\Phi^{ub}, \psi^{ub}, \chi^{ub})$.

The scheme may be summarized as follows:

1. To define balance, calculate q from model data and use (34) and (35) as constraints
2. To define imbalance, calculate δ_t and ε_t from model data using (34) and (35), and use $q = 0$ as a constraint

4 References

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