Some aspects of the HARMONIE limited-area model

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Outlook

- Vertical discretization using finite elements
- Spectral discretization in the horizontal
  - Spectral basis
  - Biperiodization
  - Relaxation to the nesting model
  - Application of relaxation and biperiodization in spectral space
- Elimination of the extension zone from the grid-point representation and increase of the width of the extension zone

Vertical discretization using finite elements (F.E.)

- In the hydrostatic version the only vertical operator is the integral
- In the non-hydrostatic version both the integral and the derivative are needed
  - This introduces some constraints when arriving at a Helmholtz equation
  - These constraints are not fulfilled by the F.E. operators

Construction of a vertical operator

$$F = \frac{df}{d\eta}$$

Derivative operator

$$F(\eta) \sim \sum_{i=1}^{M} F_i E^i(\eta)$$

$$f(\eta) \sim \sum_{i=1}^{N} f_i e^i(\eta)$$

Approximate functions as linear combinations of basis functions

$$\sum_{i=1}^{M} F_i E^i(\eta) \approx \sum_{j=1}^{N} f_j \frac{d}{d\eta} e^j(\eta)$$

Galerkin procedure

Scalarly multiply by a set of test functions

$$\underbrace{\sum_{i=1}^{M} F_i \int_0^1 E^i(\eta) T_k(\eta) d\eta}_{} = \sum_{j=1}^{N} f_j \int_0^1 \frac{d}{d\eta} e^j(\eta) T_k(\eta) d\eta \qquad \forall k \in (1-K)$$

A^i_k
(mass matrix)

B^j_k
(operator matrix)

Approximation error: orthogonal to space spanned by test functions *T*

$$\sum_{i=1}^{M} F_i A_k^i = \sum_{j=1}^{N} f_j B_k^j \Rightarrow \tilde{F}\mathbf{A} = \tilde{f}\mathbf{B}$$

K equations
M unknowns

Galerkin procedure (cont)

$\tilde{f}$ is the set of coefficients for the representation of function $f(\eta)$

If we are given the values $f(\eta_j)$ at a set of values of η (full level values)

$$f(\eta_j) = \sum_{i=1}^{M} f_i e^i(\eta_j) \equiv \tilde{f}\mathbf{P}$$

$$\tilde{f} = f(\eta_j)\mathbf{P}^{-1}$$

$\mathbf{P}^{-1}$ is the projection matrix to the space spanned by the basis functions *e*

Galerkin procedure (cont)

From the vector of values $\tilde{F}$

We can get the values of the function at full levels

$$F(\eta_l) = \sum_{j=1}^{N} F_j E^j(\eta_l) \equiv \tilde{F}\mathbf{S}$$

Where $\mathbf{S}$ Is the inverse projection matrix from the space spanned by the basis *E*

$$F(\eta_j) = \tilde{F}\mathbf{S} = \tilde{f}\mathbf{B}\mathbf{A}^{-1}\mathbf{S} = f(\eta_j)\mathbf{P}^{-1}\mathbf{B}\mathbf{A}^{-1}\mathbf{S} \equiv f(\eta_j)\mathbf{M}$$

Vertical operators (cont)

- Matrix **M** applied to the set of full-level values of field *f gives the set of full-level values of its derivative*

- *Similarly we can compute the matrix for the integral operator:* **N**

- The order of accuracy of both **M** and **N,** using cubic basis functions can be shown to be 8

- **M** and **N** are NOT the inverse of each other

Equations

$$\frac{d\mathbf{V}}{dt} + \frac{RT}{p}\nabla_\eta p + \frac{1}{m}\frac{\partial p}{\partial \eta}\nabla_\eta \phi = \varsigma$$

$$\gamma \frac{dw}{dt} + g\left(1 - \frac{1}{m}\frac{\partial p}{\partial \eta}\right) = \gamma \Omega$$

$$\frac{\partial m}{\partial t} + \nabla_\eta (m\mathbf{V}) + \frac{\partial}{\partial \eta}(m\dot{\eta}) = 0$$

$$\frac{dT}{dt} - \frac{RT}{C_p}\frac{1}{p}\frac{dp}{dt} = \frac{Q}{C_p}$$

$$\frac{dp}{dt} + \frac{C_p}{C_v} p D_3 = \frac{Qp}{C_v T}$$

$$\frac{d\phi}{dt} = gw$$

$$\frac{\partial \phi}{\partial \pi} = -m\frac{RT}{p}$$

Pressure departure and Vertical divergence

$$P = \frac{p - \pi}{\pi}$$

$$d = -g \frac{\rho}{m} \frac{\partial w}{\partial \eta}$$

The corresponding equations are

$$\frac{dP}{dt} = (1+P)\left(\frac{1}{p}\frac{dp}{dt} - \frac{1}{\pi}\frac{d\pi}{dt}\right) = -(1+P)\left(\frac{C_p}{C_v} D_3 + \frac{\dot{\pi}}{\pi}\right) + (1+P)\frac{Q}{C_v T}$$

$$\frac{d\mathrm{d}}{dt} = \mathrm{d}\frac{1}{p}\frac{dp}{dt} - \mathrm{d}\frac{1}{T}\frac{dT}{dt} - \mathrm{d}\frac{1}{m}\frac{dm}{dt} - g\frac{p}{mRT}\frac{d}{dt}\left(\frac{\partial w}{\partial \eta}\right)$$

Helmholtz equation

Eliminating from the discretized set of equations (with some constraints to be fulfilled by the operators) all the variables except the vertical divergence, we obtain a Helmholtz equation:

$$\left[1-(\Delta t)^2 c_*^2\left(m_*^2\nabla^2+\frac{\mathbf{L}^*}{rH_*^2}\right)-(\Delta t)^4\frac{N_*^2c_*^2}{r}m_*^2\nabla^2T^*\right]\mathrm{d}=r.h.s.$$

Which can be solved very easily in spectral space
In a projection on vertical eigenvectors

Choices to apply VFE in the NH version

- Choose a set of equations using only one vertical operator
  - Change the set of forecast fields
  - Change the vertical coordinate to one based on height instead of mass
- Solve a set of two coupled equations instead of a single Helmholtz equation

Change of the vertical coordinate to a height-based hybrid one

- Use of a time-independent coordinate eliminates the X-term.
- Only derivatives are used in the vertical (no integrals) which simplifies the *constraints* to arrive at a single Helmholtz equation
- The coordinate is still a hybrid coordinate. The data flow is maintained.

Change the vertical coordinate

- Juan Simarro has tested this option.
- Any vertical discretization, either finite differences or finite elements of accuracy order greater than 4 becomes unstable

Note: In general higher accuracy leads to lower stability

Solve a coupled system of equations
(Jozef Vivoda & Petra Smolikova)

- In order to arrive at a single Helmholtz equation, the following constraint (C1) has to be fulfilled

$$A_1 \equiv G^* S^* - S^* - G^* + N^* = 0$$

Where

$$\left(G^* \psi\right)_l \equiv \int_\eta^1 \frac{m^*}{\pi^*} \psi d\eta$$

$$\left(S^* \psi\right)_l \equiv \frac{1}{\pi_l^*} \int_0^{\eta_l} m^* \psi d\eta$$

$$\left(N^* \psi\right)_l \equiv \left(S^* \psi\right)_{L+1}$$

As this constraint is not fulfilled with the finite-elements integral operator, we cannot arrive at a single Helmholtz equation

Solve a coupled system of equations (cont)

Instead, we arrive at a coupled system involving both
The horizontal and the vertical divergences

$$\begin{pmatrix} \mathbb{E} & -\mathbb{F} \\ -\mathbb{B} & \mathbb{A}+\mathbb{C} \end{pmatrix} \begin{pmatrix} d \\ D \end{pmatrix} = \begin{pmatrix} d^\bullet \\ D^\bullet \end{pmatrix} .$$

where

$$\begin{aligned}
\mathbb{A} &= \left(1 - \delta t^2 c^2 \Delta\right), \\
\mathbb{B} &= \delta t^2 \Delta \left(-RT^* \mathcal{G}^* + c^2\right), \\
\mathbb{C} &= \delta t^2 \Delta R T^* A_1, \\
\mathbb{E} &= \left(1 - \delta t^2 c^2 \frac{\mathcal{L}^*}{rH^2}\right), \\
\mathbb{F} &= \delta t^2 \frac{\mathcal{L}^*}{rH^2} \left(-RT^* \mathcal{S}^* + c^2\right).
\end{aligned}$$

$$\mathcal{L}^* \psi = \frac{1}{m^*} \frac{\partial}{\partial \eta} \left(\frac{\pi^{*2}}{m^*}\right) \frac{\partial \psi}{\partial \eta} + \left(\frac{\pi^*}{m^*}\right)^2 \frac{\partial^2 \psi}{\partial \eta^2}$$

Solve a coupled system of equations (cont)

- The system of equations is twice as large as in the hydrostatic case
- An iterative procedure has been adopted for solving the system
- This method is being implemented in both HARMONIE and IFS

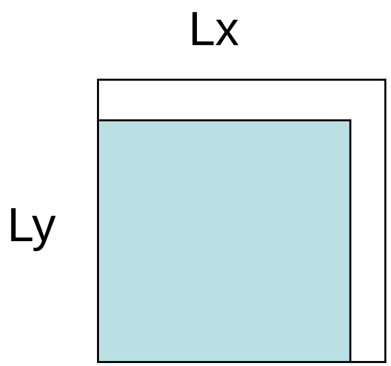
Spectral horizontal discretization

- Spherical harmonics are not an appropriate basis for a limited-area domain
- The model equations are solved on a plane projection with Cartesian x-y coordinates
- Double Fourier functions are used as the basis for spectral discretization
- Fields should be periodic in both x and y
- An extension zone is used to biperiodize the fields

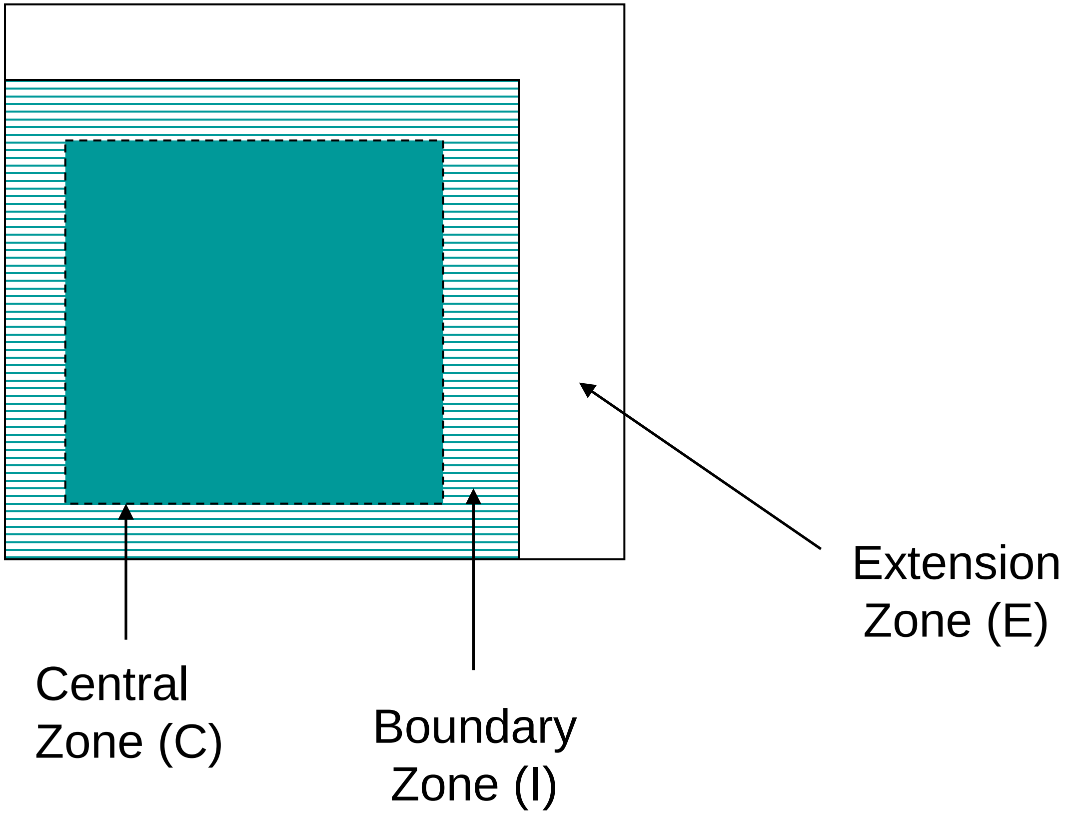
Biperiodization of fields

$$F(x,y) \approx \sum_{i=-I}^{I} \sum_{j=-J}^{J} f_k^l e^{ikx/L_x} e^{jly/L_y}$$

Periodic in x (period Lx) and in y (period Ly)



Boundary conditions



Boundary conditions
Gabor Radnoti 1995

Semi-implicit solution procedure:

$$(I - \Delta t \mathscr{L})\Psi_{t+\Delta t} = \underbrace{\Psi_{t+\Delta t(\exp)} + \Delta t \mathscr{L}\left(\Psi_{t-\Delta t} - 2\Psi_t\right)}_{\tilde{\Psi}}$$

Coupling to a nesting model (LS)

$$\Psi^C = (1-\alpha)\cdot\Psi^l + \alpha\cdot\Psi^{LS}$$

α =1 at the whole of E.
α =0 at the whole of C
Smoothly changing at I

Implementation:

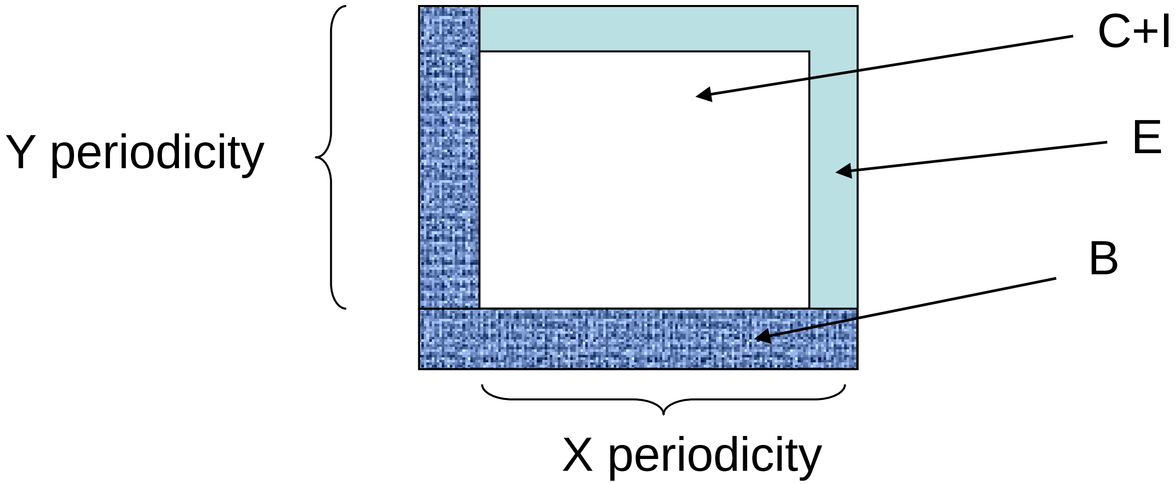
$$(I - \Delta t \mathscr{L})\Psi_{t+\Delta t} = (1-\alpha)\Psi^l + \alpha(I - \Delta t \mathscr{L})\Psi^{LS}_{t+\Delta t}$$

Boundary cond. (cont)

$\Psi_{t+\Delta t}^{LS}$ Are values derived from the nesting model.

Their values at the right border of E should join smoothly with their values at the left border of I

They can be computed by means of smoothed splines
Or by Boyd's linear combination of the values at E and at B



Increasing the width of E

- In data assimilation the influence of an observation covers an area around the observation position
- Due to the periodicity of fields, an observation close to the right border of the inner domain can affect the fields on the left border.
- That can be eliminated by increasing the width of the extension zone

Increasing the width of E (cont)

- If the points in the extension zone are present in the grid-point representation
  - The cost of running the model increases if we increase the width of E
  - Due to the clipping of the semi-Lagrangian trajectories to the C+I area, the interpolation points could fall outside the semi-Lagrangian buffer, producing floating-point errors or segmentation faults
- Elimination of the extension zone from the grid-point representation
  - Application of the boundary conditions and biperiodization in spectral space