Moving meshes over orography

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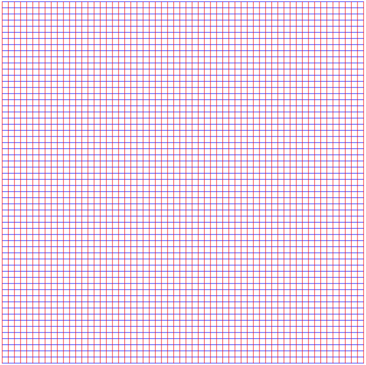
Why r-adaptivity

- Does not create load balancing problems on parallel computers,
- Does not require mapping solutions between different meshes,
- Does not necessarily lead to sudden changes in resolution,
- Can be retro-fitted into existing models

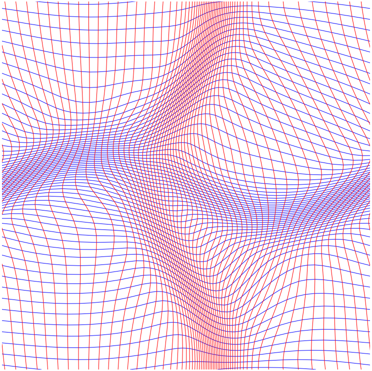
Objectives of the NERC project

- Solve optimal transport equations on the sphere to efficiently redistribute a mesh
- Assess mesh quality for the equations of the atmosphere
- Develop mimetic finite element/volume methods on moving meshes
- Compare with established test cases
- Establish suitable refinement criteria for the atmosphere

r-adaptive mesh redistribution



Original computational mesh $\mathcal{T}_c$



Adapted physical mesh $\mathcal{T}_p$

$$F(\mathcal{T}_c) = \mathcal{T}_p; \qquad \forall \xi \in \mathcal{T}_c\ \exists x \in \mathcal{T}_p \text{ s.t. } x = F(\xi) \tag{1}$$

Monitor function equidistribution

Given $m(x) > 0$, find $F : \Omega_c \to \Omega_p$ such that

$$m(x)|J(\xi)| = c. \tag{2}$$

Optimally transported meshes

Seek F^* such that

$$F^* = \arg\min_F ||F - I|| = \int_{\Omega_c} |\xi - F(\xi)|^2 \, \mathrm{d}\xi. \qquad (3)$$

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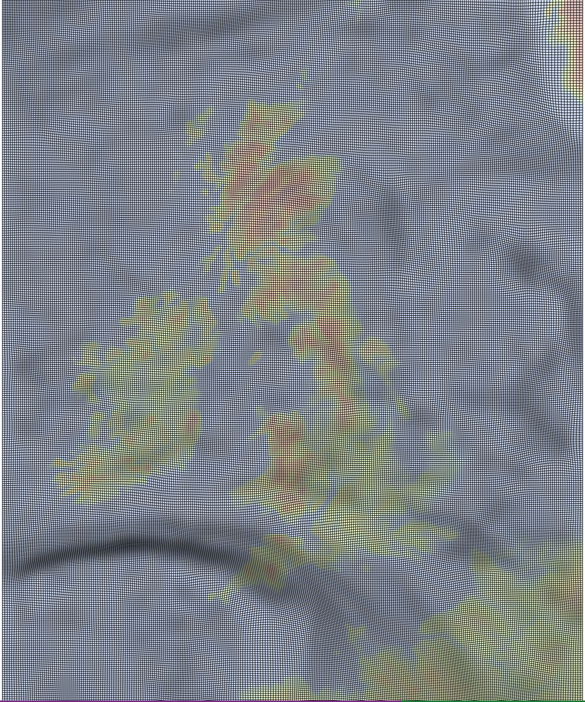
Theorem (Brenier (1991) [in cuboid domains])

There exists a **unique** *optimally transported map* $\mathbf{F}(\xi)$ *which minimises* (3)*, and the Jacobian of which satisfies the equidistribution equation (2). Furthermore,* $\mathbf{F}(\xi)$ *can be written as the gradient (with respect to* ξ*) of a convex scalar (mesh) potential* $\phi(\xi)$*, so that*

$$x(\xi) = \nabla_\xi \phi(\xi), \qquad H_\xi(\phi(\xi)) \succ 0. \tag{4}$$

Brenier, Y. (1991). Polar Factorization and Monotone Rearrangement of Vector-Valued Functions.

Communications on Pure and Applied Mathematics, XLIV:375–417



$$m(x)|H(\phi)| = c$$
$$m(\nabla\phi)|H(\phi)| = c$$

Theorem (McCann (2001))

Let M be a connected, complete smooth Riemannian manifold, equipped with its standard volume measure dx. Let μ, ν be two probability measures on M with compact support, and let the objective function $c(\xi, x)$ be equal to $d(\xi, \mathbf{x})^2$, where d is the geodesic distance on M. Further, assume that μ is absolutely continuous with respect to the volume measure on M. Then, there is a unique optimal transport map F where F pushes forward the measure μ onto ν. Then, (using classical optimal transport notation):

$$F_{\#}(\mu) = \nu \quad \textit{i.e.} \quad \mathbf{x} = F(\xi) = \exp_\xi[\nabla\phi(\xi)] \tag{5}$$

for some $d^2/2$-convex potential ϕ.

McCann, R. (2001). Polar factorization of maps on Riemannian manifolds.

Geometric & Functional Analysis GAFA, 11(3):589–608

Corollary (Weller, B., Budd, Cullen (2015))

There exists a unique, optimally transported mesh on the sphere that satisfies the equidistribution principle. Moreover, that mesh is defined by a c-convex scalar potential function that satisfies the Monge-Ampère type equation

$$m(\exp_\xi[\nabla\phi(\xi)])|J(\xi)| = c. \tag{6}$$

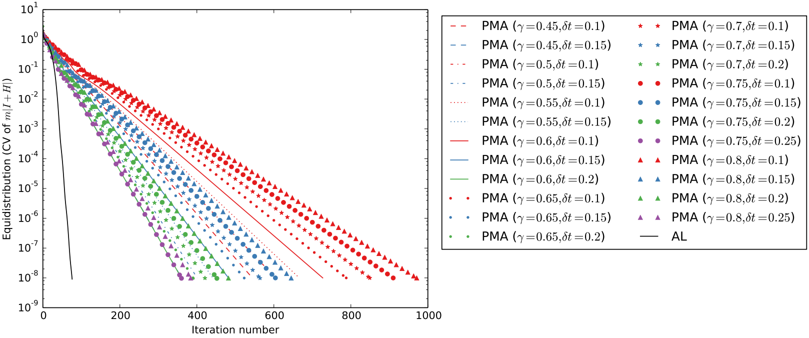
Corollary (Weller, B., Budd, Cullen (2015))

The optimally transported mesh on the sphere satisfying the equidistribution principle does not exhibit tangling.

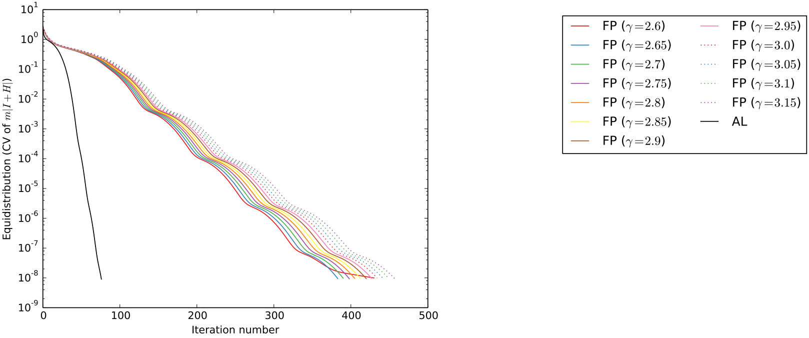
Weller, H., Browne, P., Budd, C., and Cullen, M. (2016). Mesh adaptation on the sphere using optimal transport and the numerical solution of a Monge-Ampère type equation.

Journal of Computational Physics, 308:102–123

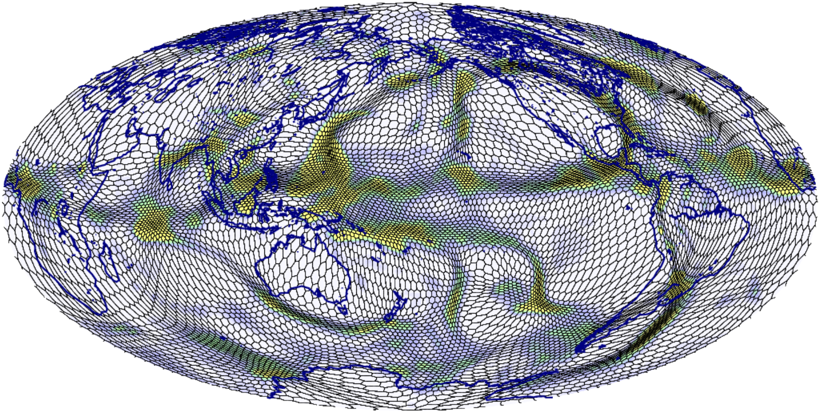
Convergence 1



Convergence 2



Mesh redistribution on the sphere



Our moving mesh strategy

- Finite volume discretisation (OpenFOAM)
- Prescribed mesh movement
  (Coming from Optimal transport solver eventually)
- No conservative mapping of fields between meshes
- Work on *"physical"* mesh and not on a computational mesh with metric terms

Moving meshes with orography

We consider

$$\frac{\partial T}{\partial t} + \nabla \cdot (\boldsymbol{u}T) = 0 \tag{7}$$

which in flux form becomes

$$\frac{\partial T}{\partial t} + \frac{1}{V}\sum_f \phi_f = 0 \tag{8}$$

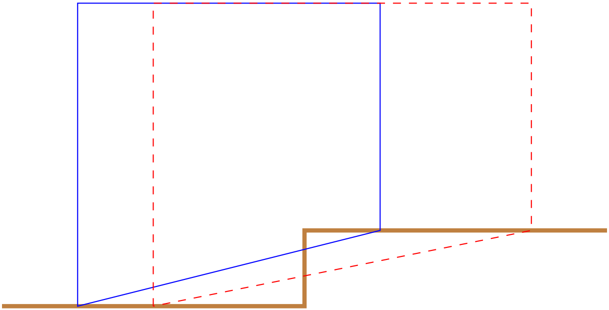
where $\phi_f = T\boldsymbol{u} \cdot \boldsymbol{n}_f$.
In the presence of a *mesh velocity* $\boldsymbol{u}^m$, the equation which we solve becomes

$$\frac{\partial T}{\partial t} + \frac{1}{V}\sum_f \left(\phi_f - \phi_f^m\right) = 0 \tag{9}$$

Look at a 1D problem, i.e. a single cell width in y and z.
Consider only u s.t. $\nabla \cdot (\boldsymbol{u}) = 0$.

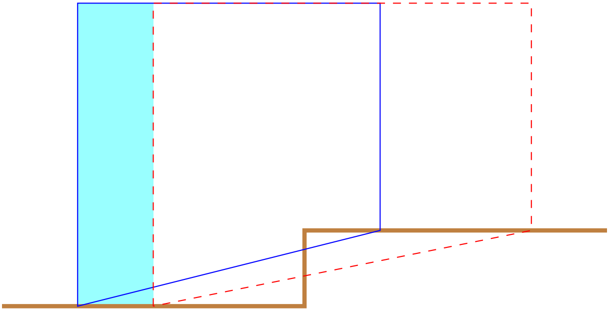
The difficulties

$$V^{t+1} = V^t + \delta t \sum_f \phi_f^m \tag{10}$$



The difficulties

$$V^{t+1} = V^t + \delta t \sum_f \phi_f^m \qquad (10)$$



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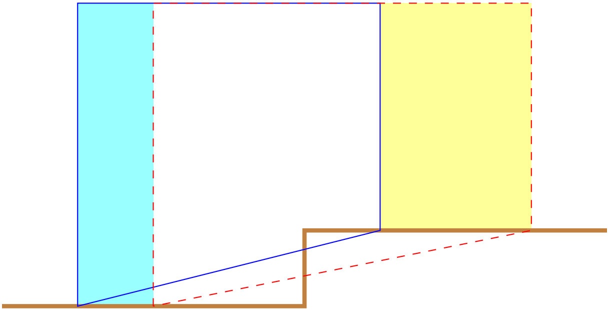

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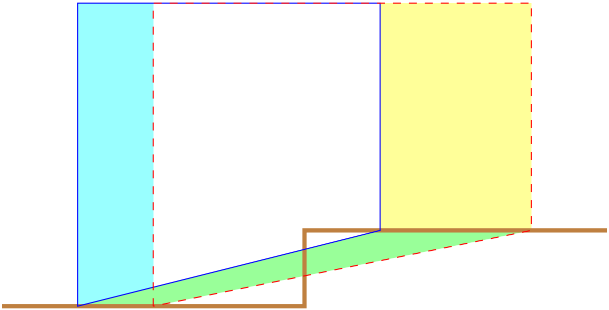
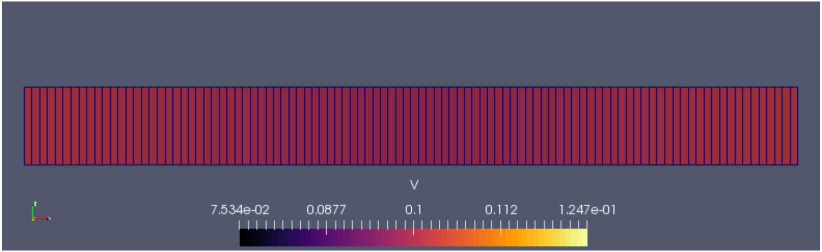
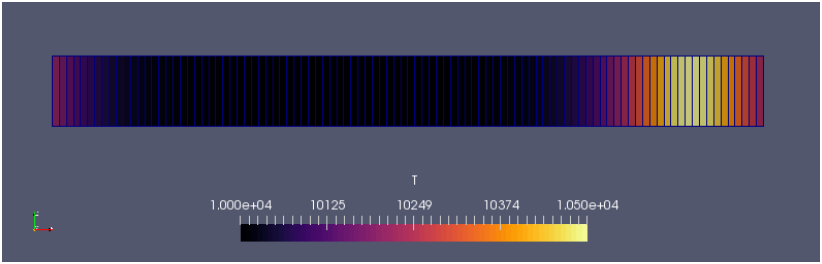
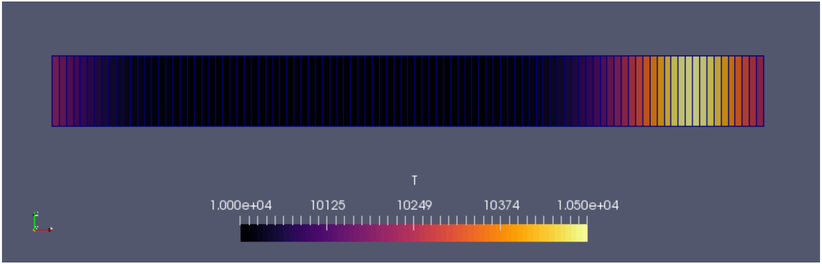


The moving mesh

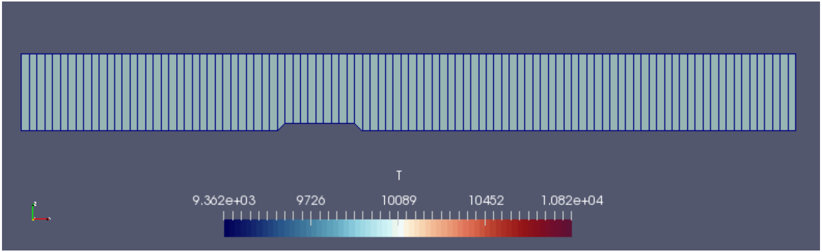


Nonconstant T, $\nabla \cdot (\boldsymbol{u}) = 0, \boldsymbol{u} \neq 0$





Orography: constant T, $\nabla \cdot (\boldsymbol{u}) = 0, \boldsymbol{u} \neq 0$ – not working!



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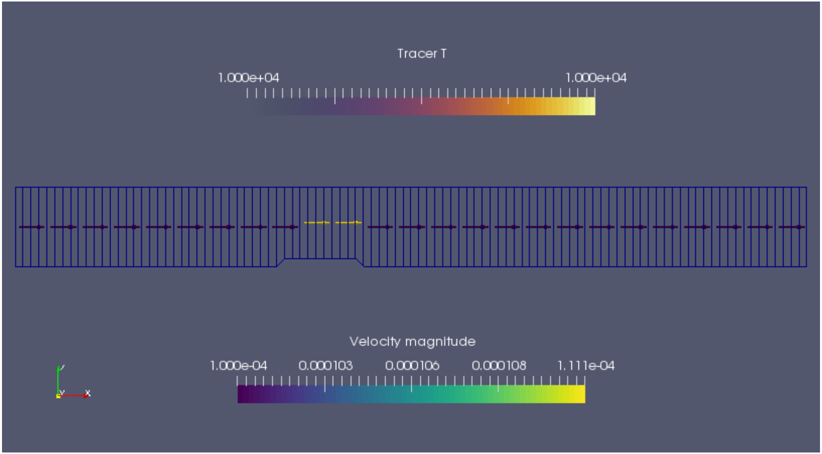
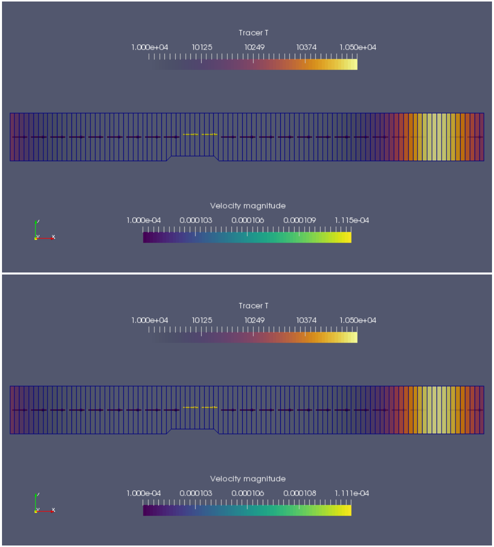
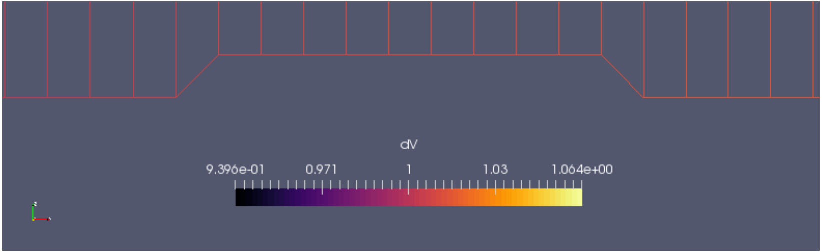


Variable T, $\nabla \cdot (\boldsymbol{u}) = 0, \boldsymbol{u} \neq 0$



Volume of the mesh



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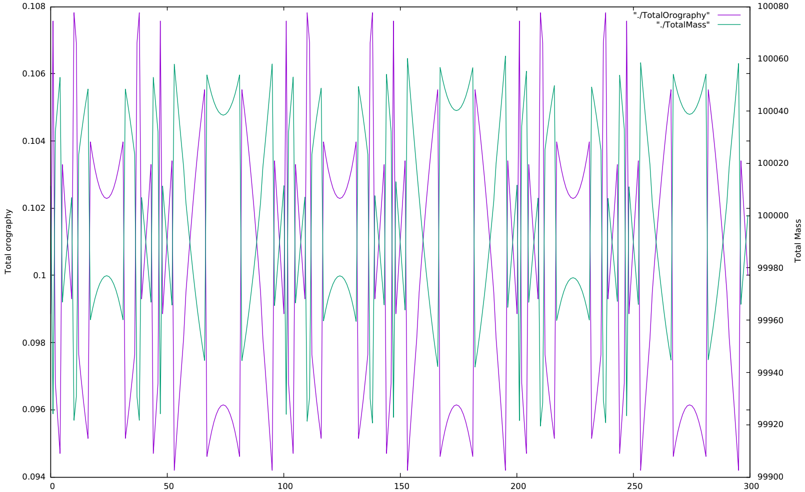

Future work

- Move from explicit to implicit timestepping
- Implement the shallow water equations - does changing mass & mesh volume introduce spurious waves?
- Extension to the a ring and then a spherical shell - does spherical geometry impact on the calculations of the mesh fluxes ϕ_f^m?
- Selection of monitor functions: is vorticity actually useful or is there a better, more robust refinement measure?

Thank you for listening

Browne, P., Budd, C., Piccolo, C., and Cullen, M. (2014). Fast three dimensional r-adaptive mesh redistribution.

Journal of Computational Physics, 275:174–196

Weller, H., Browne, P., Budd, C., and Cullen, M. (2016). Mesh adaptation on the sphere using optimal transport and the numerical solution of a Monge-Ampère type equation.

Journal of Computational Physics, 308:102–123

Browne, P., Prettyman, J., Weller, H., Pryer, T., and Lent, J. V. (2016). Nonlinear solution techniques for solving a Monge-Ampère equation for redistribution of a mesh.
http://arxiv.org/abs/1609.09646.

Under Review